Macdonald positivity conjecture

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Abstract

This is the writeup of my talk at MIT-Northeastern graduate seminar "Quantum cohomology and representation theory". This notes follow papers [6] for an introductory part about Macdonald polynomials and [1] for the proof of Macdonald positivity conjecture.

1 Statement of the positivity conjecture

1.1 Transformed Macdonald polynomials

Let Λ_R be the ring of symmetric functions of inifinetely many variables over the ring R.

For the geometric formulation of the Macdonald positivity conjecture that will follow it will be convenient to work with *transformed* Macdonald polynomials (see [6]). See Appendix for the relation between transformed and standard Macdonald and Kostka-Macdonald polynomials.

Let $S_n - \mathrm{mod}$ be the category of finite-dimensional S_n-modules, let $S_n - \mathrm{mod}_{bg}$ be the category of finite dimensional bigraged S_n-modules. Let $F : K_0(S_n - \mathrm{mod}) \to \Lambda_{\mathbb{Q}}$ be the Frobenius character map, which is defined by $F(V_\lambda) = s_\lambda$ for the irreducible representation V_λ corresponding to a Young diagram λ and where s_λ stands for the Schur polynomial. Define, abusing the notation, the Frobenius series map $F : K_0(S_n - \mathrm{mod}_{bg}) \to \Lambda_{\mathbb{Q}(q,t)}$ by $FM(q,t) = \sum_{i,j} q^i t^j FM_{i,j}$. Let $f \in \Lambda_{\mathbb{Q}(q,t)}$ be a Frobenius series of $A \in K_0(S_n - \mathrm{mod}_{bg})$. Define

$$P_x f = \sum_k (-x)^k F(A \otimes \Lambda^k V)$$

where $V = \mathbb{C}^n$ is a standard representation of S_n sitting in degree $(0,0)$ and $x \in \mathbb{Q}(q,t)$.

Definition. Transformed Macdonald polynomials $H_\lambda(q,t)$ are defined by the following properties:

1. $P_q H_\lambda \in \mathbb{Q}(q,t)\{s_\mu : \mu \geq \lambda\}$

2. $P_t H_\lambda \in \mathbb{Q}(q,t)\{s_\mu : \mu \geq \lambda^*\}$

3. Coefficient of $s_{(n)}$ in H_λ is 1.

Existence of transformed Macdonald polynomials is non-trivial and is proved, for example, in [9].

1.2 Transformed Kostka-Macdonald polynomials and positivity

Transformed Kostka-Macdonald polynomials $\tilde{K}_{\mu,\lambda} \in \mathbb{Q}(q,t)$ are defined as the coefficients of decomposition

$$H_\lambda = \sum_\mu \tilde{K}_{\mu,\lambda} s_\mu.$$

The goal of these notes is to prove

Theorem 1 (Macdonald positivity conjecture). *$\tilde{K}_{\mu,\lambda}$ are polynomials in q,t with non-negative integer coefficients.*

In the following subsection we give the geometric formulation of this conjecture, due to Haiman.

1.3 Macdonald polynomials and Procesi bundles

Let Y be the Hilbert scheme of n points on $\mathbb{C}^2 = Spec\mathbb{C}[x,y]$. Let $\pi : Y \to \mathbb{C}^{2n}/S_n$ be the standard resolution. Recall that a *Procesi bundle* $\mathcal{P}$ was constructed in Gufang's talk and is a $(\mathbb{C}^\times)^2$-equivariant vector bundle on Y satisfying the following properties:

1. $End(\mathcal{P}) = \mathbb{C}[\mathbf{x},\mathbf{y}]\#S_n$.

2. $Ext^i(\mathcal{P},\mathcal{P}) = 0$ for $i > 0$.

3. $\mathcal{P}^{S_n} = \mathcal{O}$.

Here and further $\mathbf{x},\mathbf{y}$ stand for $(x_1,...,x_n),(y_1,...,y_n)$.

Haiman's idea of a proof of the positivity conjecture was to realize transformed Macdonald polynomial H_λ as the Frobenius character of the bigraded S_n-module $\mathcal{P}_\lambda$, where $\mathcal{P}_\lambda$ stands for the fiber of $\mathcal{P}$ in the fixed point of the $(\mathbb{C}^\times)^2$-action labeled by λ.

Now property (3) from the definition of transformed Macdonald polynomials is satisfied for $F\mathcal{P}_\lambda$ automatically, as $\mathcal{P}_\lambda \simeq \mathbb{C}[S_n]$ as an S_n-module.

We now reformulate the desired properties of $\mathcal{P}_\lambda$ in a more convenient way. Let $\mathfrak{h} \subset \mathbb{C}^{2n}$ be the Lagrangian spanned by x's. Note that

$$P_q F\mathcal{P}_\lambda = F[\mathcal{P}_\lambda \leftarrow \mathcal{P}_\lambda \otimes \Lambda^1\mathfrak{h} \leftarrow \mathcal{P}_\lambda \otimes \Lambda^2\mathfrak{h} \leftarrow ...]$$

where the complex in the right hand side is the Koszul complex of the $C[\mathbf{x}]$-module P_λ. Now we need prove, in particular, that if the class of the representation V_μ apperas in the decomposition of a class of the complex $\mathcal{P}_\lambda \leftarrow \mathcal{P}_\lambda \otimes \Lambda^1\mathfrak{h} \leftarrow \mathcal{P}_\lambda \otimes \Lambda^2\mathfrak{h} \leftarrow ...$ then $\mu \geq \lambda$.

We need the following

Proposition 1. *$\mathcal{P}$ is flat over $\mathbb{C}[\boldsymbol{x}]$.*

We follow the proof by Roman Bezrukavnikov. First note that, as $\mathbb{C}[\mathbf{x}]$ is free over $\mathbb{C}[\mathbf{x}]^{S_n}$, it is enough prove flatness over $\mathbb{C}[\mathbf{x}]^{S_n}$.

In Gufang's talk $\mathcal{P}$ was constructed by lifting from the positive characteristic. If we know that $\mathcal{P}$ is flat over $F_{p^n}, p \gg 0$, then it is flat over $\mathbb{Q}_p$ and hence over $\mathbb{C}$. So it is enough to prove flatness in positive characteristic.

Note that, as we saw in Sasha's talk, the projection $Y \to \Bbbk^n/S_n$ is equidimensional and thus flat – equidimensional morphism between two smooth varieties is flat. We now show that Proposition follows from this flatness after some base changes and deformation.

Because of the equivariance it is enough to prove the flatness of the completion $\hat{\mathcal{P}}_0$ of $\mathcal{P}$ on the fiber $\pi^{-1}(0)$. Let $Fr : Y \to Y^{(1)}$ be the Frobenius morphism. Recall that Gufang constructed an Azumaya algebra $\mathbb{O}$ on $Y^{(1)}$. It splits on the completion of $\pi^{-1}(0)$ with splitting bundle $\mathcal{S}$ having the same indecomposable summands as $\mathcal{P}$. Recall that $\mathbb{O}$ is the deformation of $Fr_*\mathcal{O}_Y$. But, as mentioned above, the projection $Y \to \Bbbk^n/S_n$ is flat, so after the base change by Fr and deformation we get that $\mathcal{S}$ is flat over the completion of $\pi^{-1}(0)$, and hence $\mathcal{P}$ is.

Corollary. *Theorem 1 is equivalent to the following: there exists a Procesi bundle $\mathcal{P}$ such that if $I_\lambda \in \operatorname{supp} e_\mu \mathcal{P}/\boldsymbol{x}\mathcal{P}$ then $\mu \geq \lambda$, if $I_\lambda \in \operatorname{supp} e_\mu \mathcal{P}/\boldsymbol{y}\mathcal{P}$ then $\mu \geq \lambda^*$.*

Indeed, from the flatness it follows that the Koszul complex

$$0 \leftarrow \mathcal{P} \leftarrow \mathcal{P} \otimes \Lambda^1 \mathfrak{h} \leftarrow \mathcal{P} \otimes \Lambda^2 \mathfrak{h} \leftarrow \dots$$

is a resolution of $\mathcal{P}/\mathbf{x}\mathcal{P}$.

This is the formulation of the Macdonald positivity conjecture we will now prove.

2 Proof of the positivity conjecture

Let $H_{t,c}$ be the rational Cherednik algebra introduced in Jose's talk. Set $\Delta_{t,c}(\lambda)$ to be the Verma module of $H_{t,c}$ corresponding to the partition λ. Now note that, as it was mentioned in the Ivan's lecture, $\operatorname{supp} e_\lambda \mathcal{P}/\mathbf{x}\mathcal{P} = \operatorname{supp} \Delta_{0,0}^{loc}(\lambda)$. We include $Y = Y_0 = Hilb^n(\mathbb{C}^2)$ to the family $s : \mathbb{Y} \to \mathbb{C}$ of Calogero-Moser spaces, $s^{-1}(0) = Y$, $s^{-1}(c) = Spec(Z_{0,c}) = Y_c$, where $Z_{0,c}$ stands for the center of $H_{0,c}$. We now want to study how $\operatorname{supp} \Delta_{0,c}(\lambda)$ degenerates to $\operatorname{supp} \Delta_{0,0}^{loc}(\lambda)$.

2.1 Verma and baby Verma modules

Let T_h be the standard hyperbolic torus acting on Y_c. For a point $\lambda \in Y_c^{T_h}$ let $\mathcal{L}_c(\lambda)$ be its attracting component, that is $\mathcal{L}_c(\lambda) = \{x \in Y_c : \lim\limits_{t \in T_h, t \to \infty} t.x = \lambda\}$.

Proposition 2. $A := S(\mathfrak{h})^{S_n} \otimes S(\mathfrak{h}^*)^{S_n} \hookrightarrow Z_{0,c}$.

Recall the Dunkl embedding $\Theta_{0,c} : H_{0,c} \to \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]\#S_n$. It obviously sends $S(\mathfrak{h}^*)^{S_n}$ to the center of $\mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]\#S_n$, so to the center of $Z_{0,c}$. Now, as the definition of $H_{t,c}$ is symmetric in $\mathfrak{h}$ and $\mathfrak{h}^*$, $S(\mathfrak{h})^{S_n}$ also embeds to $Z_{0,c}$.

We get a T_h-equivariant projection $\gamma : Y_c \to \mathfrak{h}/S_n \times \mathfrak{h}^*/S_n$, so that $Y_c^{T_h} = \gamma^{-1}(0)$. Now let $A_+ \subset A$ be the ideal of polynomials without the constant term. Define

$$\tilde{H}_{0,c} = \frac{H_{0,c}}{A_+ H_{0,c}}.$$

PBW gives

$$\tilde{H}_{0,c} = \frac{S(\mathfrak{h})}{S(\mathfrak{h})_+^{S_n}} \otimes \frac{S(\mathfrak{h}^*)}{S(\mathfrak{h}^*)_+^{S_n}} \# S_n.$$

In particular, $dim\tilde{H}_{0,c} = (n!)^3$. Set $\tilde{H}_{0,c}^{-} = \dfrac{S(\mathfrak{h}^*)}{S(\mathfrak{h}^*)_+^{S_n}} \# S_n$.

Definition. Baby Verma module corresponding to the partition λ is

$$M_{0,c}(\lambda) = \tilde{H}_{0,c} \otimes_{\tilde{H}_{0,c}^{-}} V_\lambda$$

where V_λ is an irreducible representation of S_n corresponding to λ.

It is obvious from the definition that $M_{0,c}(\lambda)$ is supported on $\gamma^{-1}(0)$. Moreover, $M_{0,c}(\lambda)$ is a quotient of $\Delta_{0,c}(\lambda)$ and the latter is supported on a single attracting component of T_h-action, so $M_{0,c}(\lambda)$ is supported in a single fixed point. We will say that this fixed point corresponds to λ.

In the end of the day we get that, for $c \neq 0$, $\operatorname{supp} \Delta_{0,c}(\lambda) \subset \mathcal{L}_c(\lambda)$.

Finally we prove the

Proposition 3. $\operatorname{supp} \Delta_{0,0}^{loc}(\lambda) \subset \cup_{\mu \leq \lambda} \mathcal{L}_0(\mu)$.

Let $\mathbb{L}(\lambda) \subset \mathbb{Y}$ be a closure of the union $\cup_{c \neq 0} \mathcal{L}_c(\lambda)$. This is an irreducible variety such that its fiber over $c \neq 0$ is $\mathcal{L}_c(\lambda)$. Recall from Sasha's talk that $H_*^{BM,T_h}(Y_c) = \Lambda_{n,\mathbb{Q}(t)}$ (note that these homology groups for different c are canonically and T_h-equivariantly identified via the identification of the fixed points). Recall also that the class $[\mathcal{L}_0(\lambda)]$ corresponds to m_λ, while the class of the fixed point $[I_\lambda]$ corresponds to a polynomial proportional to s_λ. Note that, for $c \neq 0$, $\mathcal{L}_c(\lambda)$ is closed, irreducible and contracts to the fixed point, so its class in $H_*^{BM,T_h}(Y_c)$ is proportional to a point class. We get that $[\mathbb{L}_\lambda \cap Y_0]$ corresponds to a polynomial, proportional to s_λ. But $s_\lambda = \sum_{\mu \leq \lambda} c_{\lambda,\mu} m_\mu$, so Proposition follows.

3 Appendix. Macdonald polynomials

Remember that Sasha introduced Jack polynomials P_λ^α as an orthogonalization of the basis m_λ of the space $\Lambda_{\mathbb{Q}(\alpha)}$ equipped with the form given by

$$\langle p_\lambda, p_\mu \rangle_\alpha = \alpha^{l(\lambda)} z_\lambda \delta_{\lambda,\mu}$$

where $l(\lambda) = \lambda_1^*, z_\lambda = \prod l^{n_l} n_l!, \lambda = (1^{n_1}, 2^{n_2}, ...)$.

Now we define the following form on $\Lambda_{\mathbb{Q}(q,t)}$:

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} z_\lambda \delta_{\lambda,\mu}.$$

Definition. Macdonald polynomials $P_\lambda(q,t) \in \Lambda_{\mathbb{Q}(q,t)}$ are characterized by the following two properties:

1. $P_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda,\mu} m_\mu, c_{\lambda,\mu} \in \mathbb{Q}(q,t)$.

2. $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ if $\lambda \neq \mu$.

Proposition 4 ([9])**.** *Polynomials satisfying above properties exist.*

We list some straightforward properties of $P_\lambda(q,t)$.

1. $P_\lambda(q,q) = s_\lambda$.

2. $P_\lambda(0, t)$ are Hall-Littlewood polynomials.

3. $\lim_{t \to 1} P_\lambda(t^\alpha, t) = P_\lambda^{(\alpha)}$.

4. $P_\lambda(1, t) = e_{\lambda^*}$, $P_\lambda(q, 1) = m_\lambda$.

3.1 Kostka-Macdonald polynomials

Let $f \in \Lambda_{\mathbb{Q}(q,t)}$ be a Frobenius series of $A \in K_0(S_n - \mathrm{mod}_{bg})$. Define

$$Q_x f = \sum_k x^k F(A \otimes S^k V)$$

where $V = \mathbb{C}^n$ is a standard representation of S_n and $x \in \mathbb{Q}(q, t)$.

Proposition 5 ([6]). *Polynomials $J_\lambda = t^{n(\lambda)} Q_{t^{-1}} H_\lambda(q, t^{-1})$ are scalar multiples of Macdonald polynomials P_λ. Here $n(\lambda) = \sum_i (i - 1)\lambda_i$.*

Macdonald defined Kostka-Macdonald polynomials $K_{\mu,\lambda}$ as coefficients in the decomposition

$$J_\lambda = \sum_\mu K_{\mu,\lambda} Q_t s_\lambda.$$

Proposition 6. $\tilde{K}_{\mu,\lambda}(q, t) = t^{n(\lambda)} K_{\mu,\lambda}(q, t^{-1})$.

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