Quantizations of affine schemes (addendum to Lec 22).

1) Correspondence between truncated quantizations.

Let X be a finite type affine Poisson scheme, and $A=\mathbb{C}[X]$. For $k\geqslant 1$, we can talk about kth truncated quantizations of A and of $\mathcal{O}_X$, Sec. 1.1 of Lec 22. It turns out, they are in bijection. Here's one (easier) direction.

Proposition 1: Let $\mathcal{D}_{\hbar,k}$ be a kth truncated quantization of $\mathcal{O}_X$. Then $\Gamma(\mathcal{D}_{\hbar,k})$ is a kth truncated quant'n of A.

Proof: A Poisson isomorphism $\mathcal{D}_{\hbar,k}/(\hbar)\longrightarrow\mathcal{O}_X$ gives rise to a Poisson embedding $\Gamma(\mathcal{D}_{\hbar,k})/(\hbar)\hookrightarrow A$. We need to show it's an isomorphism. For this we use the long exact sequence in cohomology for the functor $R\Gamma$ applied to the SES $0\rightarrow\mathcal{D}_{\hbar,k-1}\xrightarrow{\hbar}\mathcal{D}_{\hbar,k}\longrightarrow\mathcal{O}_X\rightarrow 0$. We use $H^1(X,\mathcal{O})=0$ to deduce $H^1(\mathcal{D}_{\hbar,k})=0$ for all k. And this implies $\Gamma(\mathcal{D}_{\hbar,k})\twoheadrightarrow A$. $\square$

To get from restricted quantizations of A to those of X we need to localize. First, let's review how localization works for <u>noncommutative</u> rings.

Suppose B is a noncommutative ring and $S \subset B$ is a subset containing 1 and closed under taking products. We want the localization $B[S^{-1}]$ to satisfy the universal property and consist of right fractions as^{-1}, $a \in B$, $s \in S$. It's known that $B[S^{-1}]$ exists provided the following Ore conditions are satisfied:

O1: Every right fraction is also a left fraction: $\forall a \in B$, $s \in S$ $\exists$ $b \in B$, $t \in S$ s.t. $ta = bs$ ($\Leftrightarrow$ "$as^{-1} = t^{-1}b$"). Also, every left fraction is also a right fraction.

O2: If $s \in S$, $a \in B$ are s.t. $sa = 0$, then $\exists\, t \in S$ s.t. $at = 0$. And the other way around (for $as = 0$).

Now we get back to our situation: $B = \mathcal{A}_{\hbar,k}$, a kth truncated quantization of A. Pick $f \in A$, take its lift $\hat{f}$ to $\mathcal{A}_{\hbar,k}$ and set $S = \{\hat{f}^n, n \geq 0\}$.

Lemma: S satisfies (O1) & (O2).

Sketch of proof:

Note that $[\hat{f},\cdot]^k=0$. To establish (O1) for $s=\hat{f}$, $a\in \mathcal{A}$ we take $t=\hat{f}^k$ and a suitable b (commute $\hat{f}$ through a k times). Details as well as checking (O2) are left as exercise. $\square$

Exercise 1: Prove that every other lift $\tilde{f}$ of f to $\mathcal{A}_{\hbar,k}$ gives an invertible element in $\mathcal{A}_{\hbar,k}[S^{-1}]$. Deduce that $\mathcal{A}_{\hbar,k}[S^{-1}]$ is independent of the choice of $\hat{f}$.

So we get an algebra to be denoted by $\mathcal{A}_{\hbar,k}[f^{-1}]$.

Exercise 2: 1) Prove it's flat over $\mathbb{C}[\hbar]/(\hbar^k)$.

2) Produce a natural algebra homomorphism
$\mathcal{A}_{\hbar,k}[f^{-1}] \longrightarrow \mathcal{A}_{\hbar,k}[(fg)^{-1}]$.

Finally, we have the following result generalizing its usual commutative analog.

Proposition 2: 1) There's a unique sheaf of algebras $Loc(\mathcal{A}_{\hbar,k})$ on X s.t.

- $\Gamma(X_f, Loc(\mathcal{A}_{\hbar,k})) \xrightarrow{\sim} \mathcal{A}_{\hbar,k}[f^{-1}]$
- $\Gamma(X_f, Loc(\mathcal{A}_{\hbar,k})) \longrightarrow \Gamma(X_{fg}, Loc(\mathcal{A}_{\hbar,k}))$ coincides w. the homomorphism $\mathcal{A}_{\hbar,k}[f^{-1}] \longrightarrow \mathcal{A}_{\hbar,k}[(fg)^{-1}]$.

2) The sheaf $Loc(\mathcal{A}_{\hbar,k})$ is a kth truncated quantization of $\mathcal{O}_X$.

3) The maps $\mathcal{D}_{\hbar,k} \mapsto \Gamma(\mathcal{D}_{\hbar,k})$ are $\mathcal{A}_{\hbar,k} \mapsto Loc(\mathcal{A}_{\hbar,k})$ are mutually inverse bijections between the (isomorphism classes of) kth truncated quantizations of X and of A.

2) Correspondence between formal quantizations.

To a formal quantization $\mathcal{D}_\hbar$ on $\mathcal{O}_X$ we assign its global sections $\Gamma(\mathcal{D}_\hbar)$. This is a formal quantization of $A = \mathbb{C}[X]$. Indeed, arguing as in the proof of Prop. 1 in Sec 2, we see that $\Gamma(\mathcal{D}_{\hbar,k+1})/(\hbar^k) \xrightarrow{\sim} \Gamma(\mathcal{D}_{\hbar,k})$. Then one uses $\Gamma(\mathcal{D}_\hbar) = \Gamma(\varprojlim_k \mathcal{D}_{\hbar,k}) = \varprojlim_k \Gamma(\mathcal{D}_{\hbar,k})$ to conclude that $\Gamma(\mathcal{D}_\hbar)$ is a formal quantization of A. Details are exercise.

Now suppose we are given a formal quantization $\mathcal{A}_\hbar$ of A. Let $\mathcal{A}_{\hbar,k} = \mathcal{A}_\hbar/(\hbar^k)$. Pick $f \in A$. The universal property of localization yields a homomorphism $\mathcal{A}_{\hbar,k+1}[f^{-1}] \to \mathcal{A}_{\hbar,k}[f^{-1}]$.

Exercise: This homomorphism induces an isomorphism

$$\mathcal{A}_{\hbar,k+1}[f^{-1}]/(\hbar^k) \xrightarrow{\sim} \mathcal{A}_{\hbar,k}[f^{-1}].$$

Moreover, these homomorphisms glue together to

$$Loc(\mathcal{A}_{\hbar,k+1})/(\hbar^k) \xrightarrow{\sim} Loc(\mathcal{A}_{\hbar,k}) \qquad (*)$$

Then we can define $Loc(\mathcal{A}_\hbar)$ as $\varprojlim_k Loc(\mathcal{A}_{\hbar,k})$. It's a formal quantization of $\mathcal{O}_X$; $(*) \Rightarrow Loc(\mathcal{A}_\hbar)/(\hbar^k) \xrightarrow{\sim} Loc(\mathcal{A}_{\hbar,k})$ $\forall k$, these claims are left as exercises.

On the other hand, have $\mathcal{D}_\hbar = \varprojlim_k \mathcal{D}_{\hbar,k}$, hence $\Gamma(\mathcal{D}_\hbar) = \varprojlim_k \Gamma(\mathcal{D}_{\hbar,k})$. We have $\Gamma(\mathcal{D}_{\hbar,k+1})/(\hbar^k) \xrightarrow{\sim} \Gamma(\mathcal{D}_{\hbar,k})$, $\forall k$, hence $\Gamma(\mathcal{D}_{\hbar,k}) = \Gamma(\mathcal{D}_\hbar)/(\hbar^k)$. Now Proposition 2 from Sec 1 implies that the assignments $\mathcal{D}_\hbar \mapsto \Gamma(\mathcal{D}_\hbar)$ & $\mathcal{A}_\hbar \mapsto Loc(\mathcal{A}_\hbar)$ are mutually inverse to each other.