Basics on quantizations.

1) Filtered quantizations.

Setting: $\mathbb{F}$ base field, usually alg. closed.

A commutative unital $\mathbb{F}$-algebra equipped with:

- Poisson bracket $\{;\}$ (i.e. Lie bracket + Leibniz identity $\{a,bc\} = b\{a,c\} + c\{a,b\}$)
- Algebra grading $A = \bigoplus A_i$

compatible via $\{A_i, A_j\} \subseteq A_{i+j-1}$ (i.e. $\deg \{;\} = -1$)

Example 1: $\mathfrak{g}$ - Lie algebra, $A^g = \mathfrak{S}(\mathfrak{g})$

with standard grading & unique $\{;\}$ s.t.

$\{x,y\} = [x,y] \quad \forall x,y \in \mathfrak{g}$

Example 2: $A = \mathbb{F}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ w. grading

by degree in $y$'s & standard bracket

$\{P,Q\} = \sum_{i=1}^n \left( \frac{\partial P}{\partial y_i} \frac{\partial Q}{\partial x_i} - \frac{\partial P}{\partial x_i} \frac{\partial Q}{\partial y_i} \right)$
Example 2': This example generalizes the previous one. Let $A_0$ be $\mathbb{F}[x_0]$, the algebra of regular functions on a smooth affine variety $X_0$. Let $V := \text{Der}(A_0, A_0)$, the $A_0$-module of derivations $A_0 \to A_0$, geometrically these are vector fields on $X_0$.

Let $A_0 := S_{A_0}(V) := \mathbb{F}[T^*X_0]$ w. standard grading. $\exists \left\{ f_i \right\}$ on $A$ s.t.

\begin{align*}
\{ f g \} := 0, & \quad f, g \in A_0 \\
\{ \xi, f \} := \xi f & \quad \forall \xi \in V, f \in A_0 \\
\{ \xi, \eta \} := [\xi, \eta] & \quad \forall \xi, \eta \in V
\end{align*}

It comes from the standard symplectic form on $T^*X_0$.

In Example 2, $A_0 = \mathbb{F}[x_1, \ldots, x_n] \quad (X = \mathbb{A}^n)$ & $y_i := \frac{\partial}{\partial x_i}$ is a basis of the $A_0$-module $V$. 

[2]
Now we proceed to quantizations.

**Definitions:** Let $\mathcal{A}$ be a unital (assoc.) $\mathbb{F}$-algebra.

* A $\mathbb{Z}_{\geq 0}$-algebra filtration is collection of vector subspaces $\mathcal{A}_{\leq i} \subset \mathcal{A}$ ($i \in \mathbb{Z}_{\geq 0}$) s.t.: $-1 \in \mathcal{A}_{\leq 0}$
  - $\mathcal{A}_{\leq i} \cap \mathcal{A}_{\leq j} \subset \mathcal{A}_{\leq i+j}$, $\forall i,j$
  - $\mathcal{A} = \bigcup_{i \geq 0} \mathcal{A}_{\leq i}$

* The associated graded algebra $\text{gr} \mathcal{A} = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathcal{A}_{\leq i} / \mathcal{A}_{\leq i-1}$ - graded algebra.

* $\mathcal{A}$ is **almost commutative** if $\text{gr} \mathcal{A}$ is commutative, equivalently $[\mathcal{A}_{\leq i}, \mathcal{A}_{\leq j}] \subset \mathcal{A}_{\leq i+j}$, $\forall i,j$. In this case $\text{gr} \mathcal{A}$ acquires a Poisson bracket of deg -1: $\{a+\mathcal{A}_{\leq i-1}, b+\mathcal{A}_{\leq j-1}\} : = [a,b] + \mathcal{A}_{\leq i+j-2}$.

**Exercise:** show this indeed a Poisson bracket.
Definition: A filtered quantization of $A$ as above is a pair $(\mathcal{A}, \iota)$ of
- an almost commutative filtered algebra $\mathcal{A}$ and
- graded Poisson algebra isomorphism $\iota: A \xrightarrow{\sim} \text{gr} \mathcal{A}$.

We often omit $\iota$ when it's clear from the context.

Example 1: $\mathcal{A} = U(g) = \frac{T(g)}{(x\circ y - y\circ x - [x,y])}$

is a filtered quantization of $A = S(g) = \frac{T(g)}{(x\circ y - y\circ x)}$.

Indeed, since $T(g)$ is a graded algebra, there's a graded algebra epimorphism $T(g) \xrightarrow{\text{gr}} U(g)$ that factors through $g$. 
\[ S(g) \rightarrow \text{gr } U(g) \] is an isomorphism by the PBW Theorem.

**Exercise:** Check that \( S(g) \rightarrow \text{gr } U(g) \) intertwines the Poisson bracket.

**Example 2:** Let \( A = \mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n] \).

Set \( \mathfrak{g} = \mathbb{F} \langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle \) — free algebra.

\[
[x_i, x_j] = [\partial_i, \partial_j] = 0, \quad [\partial_i, x_j] = 1
\]

(known as the Weyl algebra). Define a filtration \( \mathfrak{g} \) by degree in \( x \)'s:

\[
\mathfrak{g}_{<i} = \text{Span} (x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \sum b_i \leq i)
\]

Similarly to the previous example, there is graded algebra epimorphism \( A \xrightarrow{\xi} \text{gr } \mathfrak{g} \).

It's Poisson.

**Exercise:** Show that \( \xi \) is an isomorphism.
by checking that the monomials
\[ x_1^{a_1} \ldots x_n^{a_n} \partial_1^{b_1} \ldots \partial_n^{b_n} \]
are linearly independent.

In order to do this consider the case

when char \( F = 0 \) and construct a representation of \( \mathfrak{S}_r \) in \( F[x_1^{a_1} \ldots x_n^{a_n}] \) showing that the images of the monomials are linearly independent.

When char \( F > 0 \), treat the case of \( \mathbb{Z} \) instead of \( F \) first & base change.

**Example 2**: Let \( A_0, V, X_0 \) be as in the previous instance of this example. Consider the associative algebra \( \mathcal{D}(X_0) \) of linear differential operators on \( X_0 \). By definition, it's generated by the algebra \( A_0 \) and left \( A_0 \)-module \( V \) (meaning that we are getting
algebra homomorphism $A_0 \to D(X)$ & a left $A$-module homomorphism $V \to D(X)$ with the following additional relations:

$\bar{f} \bar{g} = \bar{f} \bar{g} + \bar{g} \bar{f}$ \quad $\bar{f}, \bar{g} \in V, f \in A_0$

$\bar{g} \bar{f} - \bar{f} \bar{g} = [\bar{g}, \bar{f}]$ \quad $\bar{g}, \bar{f} \in V$

Consider the filtration on $D = D(X)$ by degrees in $V$. Then $\text{gr} D(X)$ is a commutative algebra with $S_0(V) \xrightarrow{\iota} \text{gr} D(X)$, a homomorphism of graded Poisson algebras. Then $\iota$ is an isomorphism. Here is a scheme of proof:

- The case of $X = \mathbb{A}^n$ - we recover Example 2.

- For an etale morphism $Y \to X$ of smooth affine varieties we have a filtered algebra structure on $\mathcal{F}[Y] \otimes_{\mathcal{F}[X]} D(X)$ & a
filtered algebra isomorphism
\[ F[y] \otimes_{F[x]} D(x) \sim D(y). \] So if the claim that \( c \) is an isomorphism holds for \( X \), then it’s also an isomorphism for \( Y \).

- Every smooth variety can be covered by “coordinate charts”: open affine sub-varieties that admit etale morphisms to affine spaces.

This finishes the proof that \( c : [F[T^*X]] \sim \text{gr } D(x) \). Hence \( D(x) \) is a filtered quantization of \( [F[T^*X]]. \)

Let us elaborate on the terminology.

**Exercise:** The actions of \( A_0 \) on itself by multiplication & of \( V \) via \( V \sim \).
\text{Der}(A_0) \text{ extend to an action of } D(X_0) \text{ on } A_0.

When char F=0 this action is faithful, compare to the proof of \( F[x_1,\ldots,x_n,y_1,\ldots,y_m] \to \text{gr } D(A^n) \) in Example 2. When char F>0, the action is NOT faithful. We will return to this later.

2) Formal quantizations.

Let \( A \) be a Poisson algebra, not necessarily graded. Let \( t \) be an independent variable.

\text{Definition: by a formal quantization of } A \text{ we mean a pair } (\mathfrak{A}_t, \phi), \text{ where}

- \( \mathfrak{A}_t \) is an associative \( \mathbb{F}[\![t]\!] \)-algebra

\( \phi \) s.t.
- $\hbar$ is not a zero divisor in $\mathcal{A}_\hbar$.

- $\mathcal{A}_\hbar$ is complete & separated in $\hbar$-adic topology, equivalently the natural homomorphism $\mathcal{A}_\hbar \to \lim \frac{\mathcal{A}_\hbar}{\hbar^n \mathcal{A}_\hbar}$ an isomorphism.

- $[\mathcal{A}_\hbar, \mathcal{A}_\hbar] \subset \hbar \mathcal{A}_\hbar$.

- $\chi : \frac{\mathcal{A}_\hbar}{\hbar \mathcal{A}_\hbar} \xrightarrow{\sim} A$ is an algebra isomorphism s.t.

$$\chi\left(\frac{1}{\hbar}[a,b]\right) = \{\chi(a),\chi(b)\} \forall a, b \in \mathcal{A}_\hbar.$$

We are not going to give separate examples of formal quantizations. Rather we'll explain how to pass from filtered quantizations to formal ones.

Let $A$ be a $\mathbb{Z}_{\geq 0}$-graded Poisson algebra $\mathfrak{P} = \bigcup_{i \geq 0} \mathfrak{P}_i$ be its filtered quantization.
**Definition:** Let $A = \bigcup_{i} K_{i}$ be a $\mathbb{Z}$-filtered algebra. The Rees algebra of $A$, $R_{h}(A)$, is defined as the subspace

$$\bigoplus_{i} h^{i} A \subset A[[h]]$$

**Exercise:** *Show that $R_{h}(A)$ is a graded subalgebra in $A[[h]]$.*

*Construct natural isomorphisms

$$R_{h}(A)/(h) \xrightarrow{\sim} \text{gr } A, R_{h}(A)/(h-1) \xrightarrow{\sim} A.$$

Then we can form the completion $\hat{R}_{h}(A)$ in the $h$-adic topology:

$$\hat{R}_{h}(A) = \lim_{\leftarrow n} R_{h}(A)/h^{n} R_{h}(A).$$

**Note that if $A$ is $\mathbb{Z}_{\geq 0}$-filtered, then**

$$\hat{R}_{h}(A) = \bigcap_{i} h^{i} A \subset A[[h]]$$

**Exercise:** *Suppose $A$ is a filtered quantization of $A$, then $\hat{R}_{h}(A)$ is a formal quantization.*
Example: Let \( \mathfrak{g} = U(\mathfrak{g}) \). Then 
\[
\hat{R}_h(\mathfrak{g}) = \frac{T(\mathfrak{g})[\hbar]}{(x \otimes y - y \otimes x - \hbar [x, y])} \quad \text{and} \quad \check{R}_h(\mathfrak{g}) = \frac{T(\mathfrak{g})[[\hbar]]}{(x \otimes y - y \otimes x - \hbar [x, y])}
\]

Under certain conditions we can also go back getting a filtered quantization from a formal one, \( \mathfrak{g}_h \).

Definition: Suppose \( A \) is \( \mathbb{Z}_{\geq 0} \)-graded. By a grading on \( \mathfrak{g}_h \) we mean an action of \( \mathbb{F}^x \) on \( \mathfrak{g}_h \) by \( \mathbb{F} \)-algebra automorphisms s.t.:

- \( t \cdot \hbar = t \hbar \) \( \forall t \in \mathbb{F}^x \)
- The action of \( \mathbb{F}^x \) on \( \mathfrak{g}_h / \hbar^* \mathfrak{g}_h \) comes from an algebra grading, equivalently, is rational.

\[ \text{The isomorphism } c: A \to \mathfrak{g}_h / \hbar^* \mathfrak{g}_h \text{ is graded.} \]
Note that $\hat{R}_h^\flat(\mathfrak{A})$ has a natural grading; the corresponding $\mathcal{F}^\times$-action is extended from the action of $R_h^\flat(\mathfrak{A})$ coming from the grading.

Now suppose we have a formal quantization $\mathfrak{A}_h^\flat$ with a grading. By its finite part, we mean the subspace of $\mathfrak{A}_h^\flat$ consisting of all elements that lie in a finite dimensional $\mathcal{F}^\times$-stable subspace. Denote it by $\mathfrak{A}_h^{\flat,\text{fin}}$.

**Example:** For $\mathfrak{A}_h^\flat = \mathcal{F}[[\hbar]]$, we have $\mathfrak{A}_h^{\flat,\text{fin}} = \mathcal{F}[[\hbar]]$.

**Exercise:**

- $\mathfrak{A}_h^{\flat,\text{fin}}$ is a $\mathcal{F}[[\hbar]]$-subalgebra of $\mathfrak{A}_h^\flat$. It's $\mathcal{F}^\times$-stable and the action of $\mathcal{F}^\times$ is rational giving rise to a grading.
- Set $\mathfrak{A}_h^\flat = \mathfrak{A}_h^{\flat,\text{fin}} / (\hbar-1)\mathfrak{A}_h^{\flat,\text{fin}}$ and equip it with a filtration induced from a grading.
Show that $\mathcal{A}$ is a filtered quantization of $A$.

Finally, show that the assignments

$\mathcal{A} \mapsto \mathcal{R}(\mathcal{A}) \& \mathcal{A} \mapsto \mathcal{R}/(\mathcal{h}-1)\mathcal{R}$

define mutually inverse bijections between the filtered quantizations and formal quantizations equipped with a grading.

**Rem:** The definition of a grading on a formal quantization works even if we only have a $\mathbb{Z}$-grading on $A$, not a $\mathbb{Z}_0$-grading. We still have a bijection between filtered & graded formal quantizations. But we need to modify the definition of the former by requiring that we have a $\mathbb{Z}$-filtration on $A$ and that filtration is complete & separated meaning
\[ \mathcal{B} \xrightarrow{\sim} \lim_{n \to -\infty} \mathcal{B}/\mathcal{B} \]

### 3) Quantizations of schemes

We will also be interested in a more general setting of quantizing more general schemes (the case of algebras corresponds to affine schemes). Here is a special case:

**Example:** Let \( X_0 \) be a smooth variety. Set \( X = T^*X_0 \). This is a symplectic algebraic variety so its structure sheaf acquires a Poisson bracket. Note that an open affine cover \( X_0 = \bigcup X^i \) gives rise to an open affine cover \( X = \bigcup X_i \), where \( X_i = T^*X^i_0 \). The Poisson bracket on \( O_X \) is glued from the brackets on \( \mathcal{O}[X^i] \).
Let $\pi: X = T^* X_0 \rightarrow X_0$ denote the projection. Then $\pi_* \mathcal{O}_X$ is a sheaf of graded Poisson algebras on $X_0$. It has a filtered quantization, the sheaf $\mathcal{D}_{X_0}$ of linear differential operators.

In general, we are interested in a Poisson scheme $X$ ("Poisson" means that the structure sheaf $\mathcal{O}_X$ is equipped with a Poisson bracket; for example every algebraic symplectic variety is Poisson). The easiest thing to define is a formal quantization of $X$, i.e. of $\mathcal{O}_X$. This just repeats the definition of a formal quantization of a Poisson algebra, where all algebras are replaced with sheaves. We
would like to emphasize that a formal quantization, $\mathcal{D}_f$, of $X$ is NOT a (quasi) coherent sheaf on $X$—indeed, it’s not even a sheaf of $\mathcal{O}_X$-modules. A good way to think about $\mathcal{D}_f$ is that it’s glued from formal quantizations of open affine subschemes.

In the case when $X$ comes with a compatible $\mathbb{C}^*$-action (meaning that $\{i; \bar{z}\}$ gets rescaled by the character $t \mapsto t^{-1}$) we can define a grading on a quantization just as before—we just need to define a rational $\mathbb{C}^*$-action on $\mathcal{D}_f/\hbar^0\mathcal{D}_f$. The easiest situation is when $X$ can be covered by $\mathbb{C}^*$-stable open affine subvarieties, $X = \bigcup X_i$. This is always the case when $X$ is normal. Here we say that
the \( \mathbb{C}^\times \)-action on \( \mathcal{D}_x / \hbar^n \mathcal{D}_x \) is rational if the actions on \( \Gamma(X, \mathcal{D}_x / \hbar^n \mathcal{D}_x) \) are rational. Then one can talk about filtered quantizations of \( X \) (at least, when \( X \) can be covered by \( \mathbb{C}^\times \)-stable open affines), the procedure of going from a formal quantization with a grading to a filtered one repeats that for quantizations of algebras. We will not need this: we will be interested in the case of \( \text{char } \mathbb{F} > 0 \), where interesting (filtered) quantizations of \( X \) are coherent sheaves on a related scheme \( X^{(1)} \).