Basics on quantizations.

1) Filtered quantizations.

Setting: $\mathbb{F}$ base field, usually alg. closed.

A comm've unital $\mathbb{F}$-algebra equipped w:

- Poisson bracket $\{\cdot,\cdot\}$ (i.e. Lie bracket + Leibniz identity $\{a,bc\}=b\{a,c\}+c\{a,b\}$)
- & algebra grading $A=\bigoplus_{i=0}^{\infty} A_i$.

compatible via $\{A_i, A_j\}\subset A_{i+j-1}$ (i.e. $\deg\{\cdot,\cdot\}=-1$)

Example 1: $\mathfrak{g}$ - Lie algebra, $A:=S(\mathfrak{g})$ w. standard grading & unique $\{\cdot,\cdot\}$ s.t. $\{x,y\}=[x,y]$ $\forall x,y\in\mathfrak{g}$.

Example 2: $A=\mathbb{F}[x_1,\ldots x_n, y_1,\ldots y_n]$ w. grading by degree in y's & standard bracket

$$\{P,Q\}:=\sum_{i=1}^{n}\left(\frac{\partial P}{\partial y_i}\frac{\partial Q}{\partial x_i}-\frac{\partial P}{\partial x_i}\frac{\partial Q}{\partial y_i}\right)$$

Example 2': This example generalizes the previous one. Let A_0 be $\mathbb{F}[X_0]$, the algebra of regular functions on a smooth affine variety X_0. Let $V:=\mathrm{Der}(A_0,A_0)$, the A_0-module of derivations $A_0\to A_0$, geometrically these are vector fields on X_0.

Set $A:=S_{A_0}(V)\,(=\mathbb{F}[T^*X_0])$ w. standard grading. $\exists!$ $\{\cdot,\cdot\}$ on A s.t.

$$\{f,g\}:=0,\quad f,g\in A_0$$
$$\{\xi,f\}:=\xi.f\quad \forall\ \xi\in V,\ f\in A_0$$
$$\{\xi,\eta\}:=[\xi,\eta]\quad \forall\ \xi,\eta\in V.$$

It comes from the standard symplectic form on T^*X_0.

In Example 2, $A_0=\mathbb{F}[x_1,\dots x_n]$ $(X=\mathbb{A}^n)$ & $y_i=\frac{\partial}{\partial x_i}$ is a basis of the A_0-module V.

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Now we proceed to quantizations.

Definitions: Let $\mathcal{A}$ be a unital (assoc.) $\mathbb{F}$-alg'a.

- A $\mathbb{Z}_{\geqslant 0}$-algebra filtration is collection of vector subspaces $\mathcal{A}_{\leqslant i} \subset \mathcal{A}$ $(i \in \mathbb{Z}_{\geqslant 0})$ s.t.:
  - $1 \in \mathcal{A}_{\leqslant 0}$
  - $\mathcal{A}_{\leqslant i} \mathcal{A}_{\leqslant j} \subset \mathcal{A}_{\leqslant i+j} \quad \forall i,j$
  - $\mathcal{A} = \bigcup_{i \geqslant 0} \mathcal{A}_{\leqslant i}$
- The associated graded algebra

$$\operatorname{gr}\mathcal{A} := \bigoplus_i \mathcal{A}_{\leqslant i} / \mathcal{A}_{\leqslant i-1} \text{ - graded algebra.}$$

- $\mathcal{A}$ is almost commutative if $\operatorname{gr}\mathcal{A}$ is commutative, equivalently $[\mathcal{A}_{\leqslant i}, \mathcal{A}_{\leqslant j}] \subset \mathcal{A}_{\leqslant i+j-1}$ $\forall i,j$. In this case $\operatorname{gr}\mathcal{A}$ acquires a Poisson bracket of deg -1:

$$\{a + \mathcal{A}_{\leqslant i-1}, b + \mathcal{A}_{\leqslant j-1}\} := [a,b] + \mathcal{A}_{\leqslant i+j-2}.$$

Exercise: show this indeed a Poisson bracket.

Definition: A filtered quantization of A as above is a pair $(\mathcal{A}, \iota)$ of

- an almost commutative filtered algebra $\mathcal{A}$ and
- graded Poisson algebra isomorphism $\iota: A \xrightarrow{\sim} \operatorname{gr}\mathcal{A}$.

We often omit ι when it's clear from the context.

Example 1: $\mathcal{A} = U(\mathfrak{g}) \left(= \dfrac{T(\mathfrak{g})}{(x\otimes y - y\otimes x - [x,y])} \right)$ is a filtered quantization of

$$A = S(\mathfrak{g}) \left(= \frac{T(\mathfrak{g})}{(x\otimes y - y\otimes x)} \right)$$

Indeed, since $T(\mathfrak{g})$ is a graded algebra, there's a graded algebra epimorphism

$T(\mathfrak{g}) \twoheadrightarrow \operatorname{gr} U(\mathfrak{g})$ that factors through

$S(\mathfrak{g})$. The resulting epimorphism $S(\mathfrak{g}) \twoheadrightarrow \operatorname{gr} \mathcal{U}(\mathfrak{g})$ is an isomorphism by the PBW Theorem.

Exercise: Check that $S(\mathfrak{g}) \xrightarrow{\sim} \operatorname{gr} \mathcal{U}(\mathfrak{g})$ intertwines the Poisson bracket.

Example 2: Let $A = \mathbb{F}[x_1, \dots x_n, y_1, \dots y_n]$.
Set $\mathcal{A} = \dfrac{\mathbb{F}\langle x_1, \dots x_n, \partial_1, \dots \partial_n \rangle}{([x_i, x_j] = [\partial_i, \partial_j] = 0, [\partial_i, x_j] = 1)}$ ← free algebra.

(known as the Weyl algebra). Define a filtration on $\mathcal{A}$ by degree in ∂'s:

$$\mathcal{A}_{\leq i} = \operatorname{Span}(x_1^{a_1} \dots x_n^{a_n} \partial_1^{b_1} \dots \partial_n^{b_n} \mid b_1 + \dots + b_n \leq i).$$

Similarly to the previous example, there is graded algebra epimorphism $A \xrightarrow{\iota} \operatorname{gr} \mathcal{A}$. It's Poisson.

5 **Exercise:** Show that ι is an isomorphism

by checking that the monomials $x_1^{a_1} \dots x_n^{a_n} \partial_1^{b_1} \dots \partial_n^{b_n}$ are linearly independent.

In order to do this consider the case when $\operatorname{char} \mathbb{F} = 0$ and construct a representation of $\mathcal{A}$ in $\mathbb{F}[x_1, \dots x_n]$ showing that the images of the monomials are linearly independent.

When $\operatorname{char} \mathbb{F} > 0$, treat the case of $\mathbb{Z}$ instead of $\mathbb{F}$ first & base change.

Example 2': Let A_0, V, X_0 be as in the previous instance of this example. Consider the associative algebra $\mathcal{D}(X_0)$ of linear differential operators on X_0. By definition, it's generated by the algebra A_0 and left A_0-module V (meaning that we are getting algebra homomorphism $A_0 \to \mathcal{D}(X_0)$ & a left A-module homomorphism $V \to \mathcal{D}(X_0)$) with the following additional relations:

$$\xi f = f\xi + \xi . f \quad \xi \in V,\ f \in A_0$$

$$\xi\eta - \eta\xi = [\xi,\eta] \quad \xi,\eta \in V$$

Consider the filtration on $\mathcal{A} = \mathcal{D}(X_0)$ by degrees in V. Then $\operatorname{gr}\mathcal{D}(X_0)$ is a commutative algebra with $S_{A_0}(V) \xrightarrow{\ c\ } \operatorname{gr}\mathcal{D}(X_0)$, a homomorphism of graded Poisson algebras. Then c is an isomorphism. Here is a scheme of proof:

- The case of $X_0 = \mathbb{A}^n$ – we recover Example 2.
- For an etale morphism $Y_0 \to X_0$ of smooth affine varieties we have a filtered algebra structure on $\mathbb{F}[Y_0] \otimes_{\mathbb{F}[X_0]} \mathcal{D}(X_0)$ & a

filtered algebra isomorphism

$$\mathbb{F}[Y_0]\otimes_{\mathbb{F}[X_0]} \mathcal{D}(X_0) \xrightarrow{\sim} \mathcal{D}(Y_0).$$ So if the claim that ι is an isomorphism holds for X_0, then it's also an isomorphism for Y_0.

- Every smooth variety can be covered by "coordinate charts": open affine subvarieties that admit etale morphisms to affine spaces.

This finishes the proof that $\iota: \mathbb{F}[T^*X_0] \xrightarrow{\sim} \operatorname{gr}\mathcal{D}(X_0)$. Hence $\mathcal{D}(X_0)$ is a filtered quantization of $\mathbb{F}[T^*X_0]$.

Let us elaborate on the terminology.

Exercise: The actions of A_0 on itself by multiplication & of V via $V \xrightarrow{\sim}$

$\mathrm{Der}(A_0)$ extend to an action of $D(X_0)$ on A_0.

When $\operatorname{char}\mathbb{F}=0$ this action is faithful, compare to the proof of $\mathbb{F}[x_1,\dots,x_n,y_1,\dots,y_n] \xrightarrow{\sim} \operatorname{gr} D(\mathbb{A}^n)$ in Example 2. When $\operatorname{char}\mathbb{F}>0$, the action is NOT faithful. We will return to this later.

2) Formal quantizations.

Let A be a Poisson algebra, not necessarily graded. Let $\hbar$ be an independent variable.

Definition: by a formal quantization of A we mean a pair $(\mathcal{A}_\hbar, \iota)$, where

- $\mathcal{A}_\hbar$ is an associative $\mathbb{F}[[\hbar]]$-algebra s.t.

- $\hbar$ is not a zero divisor in $\mathcal{A}_\hbar$.
- $\mathcal{A}_\hbar$ is complete & separated in $\hbar$-adic topology, equivalently the natural homomorphism $\mathcal{A}_\hbar \longrightarrow \varprojlim_n \mathcal{A}_\hbar/\hbar^n\mathcal{A}_\hbar$ an isomorphism.
- $[\mathcal{A}_\hbar, \mathcal{A}_\hbar] \subset \hbar\mathcal{A}_\hbar$.

• $\iota: \mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar \xrightarrow{\sim} A$ is an algebra isomorphism s.t.

$$\iota\left(\frac{1}{\hbar}[a,b]\right) = \{\iota(a), \iota(b)\} \ \forall a, b \in \mathcal{A}_\hbar.$$

We are not going to give separate examples of formal quantizations. Rather we'll explain how to pass from filtered quantizations to formal ones.

Let A be a $\mathbb{Z}_{\geq 0}$-graded Poisson algebra & $\mathcal{A} = \bigcup_{i \geq 0} \mathcal{A}_{\leq i}$ be its filtered quantization.

Definition: Let $\mathcal{A} = \bigcup \mathcal{A}_{\leq i}$ be a $\mathbb{Z}$-filtered algebra. The Rees algebra of $\mathcal{A}$, $R_\hbar(\mathcal{A})$ is defined as the subspace
$$\bigoplus_i \mathcal{A}_{\leq i} \hbar^i \subset \mathcal{A}[\hbar^{\pm 1}].$$

Exercise: • Show that $R_\hbar(\mathcal{A})$ is a graded subalgebra in $\mathcal{A}[\hbar^{\pm 1}]$.

• Construct natural isomorphisms
$$R_\hbar(\mathcal{A})/(\hbar) \xrightarrow{\sim} \operatorname{gr} \mathcal{A}, \quad R_\hbar(\mathcal{A})/(\hbar - 1) \xrightarrow{\sim} \mathcal{A}.$$

Then we can form the completion $\hat{R}_\hbar(\mathcal{A})$ in the $\hbar$-adic topology:
$$\hat{R}_\hbar(\mathcal{A}) = \varprojlim R_\hbar(\mathcal{A}) / \hbar^n R_\hbar(\mathcal{A}).$$

Note that if $\mathcal{A}$ is $\mathbb{Z}_{\geqslant 0}$-filtered, then
$$\hat{R}_\hbar(\mathcal{A}) = \prod_{i \geqslant 0} \mathcal{A}_{\leq i} \hbar^i \subset \mathcal{A}[[\hbar]]$$

Exercise: Suppose $\mathcal{A}$ is a filtered quantization of A, then $\hat{R}_\hbar(\mathcal{A})$ is a formal quantization.

Example: Let $\mathcal{A} = U(\mathfrak{g})$. Then $R_\hbar(\mathcal{A}) =$

$$= \frac{T(\mathfrak{g})[\hbar]}{(x\otimes y - y\otimes x - \hbar[x,y])} \ \& \ \hat{R}_\hbar(\mathcal{A}) = \frac{T(\mathfrak{g})[[\hbar]]}{(x\otimes y - y\otimes x - \hbar[x,y])}$$

Under certain conditions we can also go back getting a filtered quantization from a formal one, $\mathcal{A}_\hbar$.

Definition: Suppose A is $\mathbb{Z}_{\geq 0}$-graded. By a grading on $\mathcal{A}_\hbar$ we mean an action of $\mathbb{F}^\times$ on $\mathcal{A}_\hbar$ by $\mathbb{F}$-algebra automorphisms s.t.:

- $t.\hbar = t\hbar \ \forall\, t \in \mathbb{F}^\times$.
- The action of $\mathbb{F}^\times$ on $\mathcal{A}_\hbar / \hbar^n \mathcal{A}_\hbar$ comes from an algebra grading, equivalently, is rational.
- The isomorphism $\iota : A \to \mathcal{A}_\hbar / \hbar \mathcal{A}_\hbar$ is graded.

Note that $\hat{R}_\hbar(\mathcal{A})$ has a natural grading: the corresponding $\mathbb{F}^\times$-action is extended from the action of $R_\hbar(\mathcal{A})$ coming from the grading.

Now suppose we have a formal quantization $\mathcal{A}_\hbar$ with a grading. By its finite part, we mean the subspace of $\mathcal{A}_\hbar$ consisting of all elements that lie in a finite dimensional $\mathbb{F}^\times$-stable subspace. Denote it by $\mathcal{A}_{\hbar,fin}$.

Example: For $\mathcal{A}_\hbar = \mathbb{F}[[\hbar]]$, we have $\mathcal{A}_{\hbar,fin} = \mathbb{F}[\hbar]$.

Exercise: • $\mathcal{A}_{\hbar,fin}$ is a $\mathbb{F}[\hbar]$-subalgebra of $\mathcal{A}_\hbar$. It's $\mathbb{F}^\times$-stable and the action of $\mathbb{F}^\times$ is rational giving rise to a grading.

• Set $\mathcal{A} := \mathcal{A}_{\hbar,fin} / (\hbar - 1)\mathcal{A}_{\hbar,fin}$ and equip it with a filtration induced from a grading.

Show that $\mathcal{A}$ is a filtered quantization of A.

- Finally, show that the assignments $\mathcal{A} \mapsto \hat{R}_\hbar(\mathcal{A})$ & $\mathcal{A}_\hbar \mapsto \mathcal{A}_{\hbar,fin}/(\hbar-1)\mathcal{A}_{\hbar,fin}$ define mutually inverse bijections between the filtered quantizations and formal quantizations equipped with a grading.

Rem: The definition of a grading on a formal quantization works even if we only have a $\mathbb{Z}$-grading on A, not a $\mathbb{Z}_{\geq 0}$-grading. We still have a bijection between filtered & graded formal quantizations. But we need to modify the definition of the former by requiring that we have a $\mathbb{Z}$-filtration on $\mathcal{A}$ and that filtration is complete & separated meaning

$$\mathcal{A} \xrightarrow{\sim} \varprojlim_{n\to-\infty} \mathcal{A}/\mathcal{A}_{\leq -n}.$$

3) Quantizations of schemes.

We will also be interested in a more general setting of quantizing more general schemes (the case of algebras corresponds to affine schemes). Here is a special case:

Example: Let X_0 be a smooth variety. Set $X = T^*X_0$. This is a symplectic algebraic variety so its structure sheaf acquires a Poisson bracket. Note that an open affine cover $X_0 = \bigcup_i X_0^i$ gives rise to an open affine cover $X = \bigcup_i X^i$, where $X^i = T^*X_0^i$. The Poisson bracket on $\mathcal{O}_X$ is glued from the brackets on $\mathbb{F}[X^i]$.

Let $\pi : X = T^*X_0 \longrightarrow X_0$ denote the projection. Then $\pi_* \mathcal{O}_X$ is a sheaf of graded Poisson algebras on X_0. It has a filtered quantization, the sheaf $\mathcal{D}_{X_0}$ of linear differential operators.

In general, we are interested in a Poisson scheme X ("Poisson" means that the structure sheaf $\mathcal{O}_X$ is equipped with a Poisson bracket; for example every algebraic symplectic variety is Poisson). The easiest thing to define is a formal quantization of X, i.e. of $\mathcal{O}_X$. This just repeats the definition of a formal quantization of a Poisson algebra, where all algebras are replaced with sheaves. We would like to emphasize that a formal quantization, $\mathcal{D}_\hbar$, of X is NOT a (quasi) coherent sheaf on X - indeed, it's not even a sheaf of $\mathcal{O}_X$-modules. A good way to think about $\mathcal{D}_\hbar$ is that it's glued from formal quantizations of open affine subschemes.

In the case when X comes with a compatible $\mathbb{C}^\times$-action (meaning that $\{\cdot,\cdot\}$ gets rescalled by the character $t \mapsto t^{-1}$) we can define a grading on a quantization just as before - we just need to define a rational $\mathbb{C}^\times$-action on $\mathcal{D}_\hbar / \hbar^n \mathcal{D}_\hbar$. The easiest situation is when X can be covered by $\mathbb{C}^\times$-stable open affine subvarieties, $X = \bigcup_i X^i$. This is always the case when X is normal. Here we say that the $\mathbb{C}^\times$-action on $\mathcal{D}_\hbar/\hbar^n\mathcal{D}_\hbar$ is rational if the actions on $\Gamma(X_i^\wedge, \mathcal{D}_\hbar/\hbar^n\mathcal{D}_\hbar)$ are rational.

Then one can talk about filtered quantizations of X (at least, when X can be covered by $\mathbb{C}^\times$-stable open affines), the procedure of going from a formal quantization with a grading to a filtered one repeats that for quantizations of algebras.

We will not need this: we will be interested in the case of char $\mathbb{F}>0$, where interesting (filtered) quantizations of X are coherent sheaves on a related scheme $X^{(1)}$.