Quantizations via Hamiltonian reduction

1) General definition
2) Examples.
3?) Quantization commutes w. reduction

Ref: Etingof.

1.1) Moment maps.

$\mathbb{F}$ is alg. closed base field

A: graded Poisson algebra, $\deg\{\cdot,\cdot\} = -1$

$\mathcal{A}$: filtered quantization.

G algebraic group

$G \curvearrowright \mathcal{A}$ rational & by filt. algebra autom'ms.

$\leftarrow$ by differentiating G-action

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \operatorname{Der}(\mathcal{A}) \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ \xi & \longmapsto & \xi_{\mathcal{A}} \end{array}$$

$\mathfrak{g} \longrightarrow \operatorname{Der}(\mathcal{A})$ – G-equiv't Lie alg. homomorphism

Def: A quantum comoment map for this action is a G-equivariant linear map $\Phi: \mathfrak{g} \longrightarrow \mathcal{A}_{\leq 1}$ s.t. $[\Phi(\xi), a] = \xi_{\mathcal{A}} a \ \forall \xi \in \mathfrak{g},\ a \in \mathcal{A}$.

Exercise: Φ is a Lie algebra homomorphism.

If $G \curvearrowright A$ rationally & by graded Poisson automorphisms can talk about (classical) comoment map:

$\varphi: \mathfrak{g} \xrightarrow[G]{} A_1$ s.t. $\{\varphi(\xi), \bullet\} = \xi_A$.

If A is fin. gen'd $\rightsquigarrow X = \operatorname{Spec}(A)$. To give $\varphi: \mathfrak{g} \to A$ ($\leftrightsquigarrow S(\mathfrak{g}) \to A$) is the same thing as to give $X \to \mathfrak{g}^*$. This is the moment map.

Rem: If $\Phi: \mathfrak{g} \to \mathcal{A}_{\leq 1}$ is quantum comoment map $\Rightarrow$ $\varphi := \operatorname{gr}\Phi$ is a classical comoment map.

Example: Let X_0 be smooth & affine, $A := \mathbb{F}[T^*X_0]$, $\mathcal{A} = \mathcal{D}(X_0)$. If $G \curvearrowright X_0 \rightsquigarrow$ rational actions $G \curvearrowright A, \mathcal{A}$ as above; $\rightsquigarrow \mathfrak{g} \to \operatorname{Vect}(X_0)$, $\xi \mapsto \xi_{X_0}$

Exercise: $\xi \mapsto \xi_{X_0}: \mathfrak{g} \to \operatorname{Vect}(X_0) \hookrightarrow \mathcal{D}(X_0)_{\leq 1}$ is a quantum comoment map, while $\mathfrak{g} \to \operatorname{Vect}(X_0) \hookrightarrow \mathbb{F}[T^*X_0]_1$ is classical comoment map.

Rem: Now let A be Poisson algebra, $\mathcal{A}_\hbar$ be its formal quantization. The defn of classical comoment map is as before but now we want $\varphi: \mathfrak{g} \to A$.

Modifications for quantum comoment map:

Assume $G \curvearrowright \mathcal{A}_\hbar$ is by $\mathbb{F}[[\hbar]]$-algebra autom'ms & $G \curvearrowright \mathcal{A}_\hbar/(\hbar^n)$ is rational $\forall n > 0$. $\rightsquigarrow \mathfrak{g} \to \operatorname{Der}_{\mathbb{F}[[\hbar]]}(\mathcal{A}_\hbar)$, $\xi \mapsto \xi_{\mathcal{A}_\hbar}$; A quantum comoment map is a G-equiv't linear $\Phi_\hbar: \mathfrak{g} \to \mathcal{A}_\hbar$ s.t. $\frac{1}{\hbar}[\Phi_\hbar(\xi), a] = \xi_{\mathcal{A}_\hbar}(a)$.

These settings are intertwined by the bijection between:

- filtered quantization
- formal quantization w. grading

(this is left as exercise).

1.2) Hamiltonian reduction

Classical Hamiltonian reduction: A is Poisson algebra, $\varphi: \mathfrak{g} \to A$ is comoment map $\rightsquigarrow$ $A\varphi(\mathfrak{g}) \subset A$ is G-stable ideal $\rightsquigarrow$ $(A/A\varphi(\mathfrak{g}))^G$ – commut. algebra.

Exercise: • Have well-defined binary operation on $(A/A\varphi(\mathfrak{g}))^G$
$\{a + A\varphi(\mathfrak{g}), b + A\varphi(\mathfrak{g})\} := \{a, b\} + A\varphi(\mathfrak{g})$ (hint: $\{\varphi(\xi), a\} = \xi_A(a)$ & if $a + A\varphi(\mathfrak{g})$ is G-invariant, then $\xi_A(a) \in A\varphi(\mathfrak{g})$).

- Get a Poisson bracket on $(A/A\varphi(\mathfrak{g}))^G$.

If A is graded & $\operatorname{im}\varphi \subset A_1 \Rightarrow (A/A\varphi(\mathfrak{g}))^G$ inherits a grading & $\deg \{\cdot,\cdot\} = -1$.

Quantum Hamiltonian reduction: A is graded, $\mathcal{A}$ is filt. quant'n. Then quantum Hamiltonian reduction is $(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{g}))^G$.

Exercise: There is well-defined associative product on $(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{g}))^G$: $(a + \mathcal{A}\Phi(\mathfrak{g}))(b + \mathcal{A}\Phi(\mathfrak{g})) := ab + \mathcal{A}\Phi(\mathfrak{g})$

Filtr'n on $\mathcal{A} \rightsquigarrow$ filtr'n on $(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{g}))^G$ w. $\deg[\cdot,\cdot] \le -1$.

Question: Is $(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{g}))^G$ a filtered quant'n of $(A/A\varphi(\mathfrak{g}))^G$?
Answer: Sometimes...

$\mathrm{gr}(\mathcal{A}\Phi(\mathfrak{g})) \supset \varphi(\mathfrak{g}) \Rightarrow$ contains $A\varphi(\mathfrak{g}) \Rightarrow$
$A/A\varphi(\mathfrak{g}) \twoheadrightarrow \mathrm{gr}(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{g}))$
$\mathrm{gr}\,[(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{g}))^G] \hookrightarrow [\mathrm{gr}(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{g}))]^G$ exercise

isomorphism when G is "linearly reductive" (all rational reps are completely reducible)

(In Part 3 will see sufficient conditions for positive answer).

Remarks: • $\chi \in (\mathfrak{g}^*)^G \rightsquigarrow \Phi_\chi := \Phi - \chi \cdot 1 : \mathfrak{g} \to \mathcal{A}_{\le 1}$. Still a quantum comoment map. Quantum Ham'n reduction gives a family of algebras param'd by χ: $(\mathcal{A}/\mathcal{A}\Phi_\chi(\mathfrak{g}))^G$.

• Have similar constr'n for formal quant'ns:
$(\mathcal{A}_\hbar/\mathcal{A}_\hbar\Phi_\hbar(\mathfrak{g}))^G$
Always have $[\mathcal{A}_\hbar/\mathcal{A}_\hbar\Phi_\hbar(\mathfrak{g})]/\hbar[\mathcal{A}_\hbar/\mathcal{A}_\hbar\Phi_\hbar(\mathfrak{g})] \xleftarrow{\sim} A/A\varphi(\mathfrak{g})$
but it can happen that $\hbar$ is a zero divisor in $\mathcal{A}_\hbar/\mathcal{A}_\hbar\Phi_\hbar(\mathfrak{g})$
so $(\mathcal{A}_\hbar/\mathcal{A}_\hbar\Phi_\hbar(\mathfrak{g}))^G$ may still fail to be quantization on $(A/A\varphi(\mathfrak{g}))^G$.

2) Examples: $\mathcal{A}=D(X_0)$, X_0 is smooth affine, $G \curvearrowright X_0$

$\Phi(\xi):=\xi_{X_0}$; $\chi\in(\mathfrak{g}^*)^G \rightsquigarrow \Phi_\chi(\xi):=\xi_{X_0}-\langle\chi,\xi\rangle$

$D(X_0)/\!/\!/_\chi G := \left(D(X_0)/D(X_0)\Phi_\chi(\mathfrak{g})\right)^G$.

2.1) $\chi=0$ & $X_0\longrightarrow Y_0$ is principal G-bundle

i.e. $\exists$ surjective etale morphism $\tilde{Y}_0\longrightarrow Y_0$ s.t.

$$\begin{tikzcd} G\times\tilde{Y}_0 \arrow[rr, "\sim"] \arrow[dr] & & \tilde{Y}_0\times_{Y_0} X \arrow[dl] \\ & \tilde{Y}_0 & \end{tikzcd}$$

commutes.

Note: can assume $\tilde{Y}_0$ is affine

Prop: $D(X_0)/\!/\!/_0 G$ is $D(Y_0)$.

Sketch of proof: $D(Y_0)$ is generated by $B_0:=\mathbb{F}[Y_0]=\mathbb{F}[X_0]^G$ & $\mathcal{U}:=\mathrm{Vect}(Y_0)$ subject to rel'ns.

Plan: • produce maps $B_0, \mathcal{U}\longrightarrow D(X_0)/\!/\!/_0 G$

• check the rel'ns for them $\rightsquigarrow D(Y_0)\longrightarrow D(X_0)/\!/\!/_0 G$

• check this is an isom'm.

Need to describe $\mathcal{U}$; $A_0=\mathbb{F}[X_0]$, $V:=\mathrm{Vect}(X_0)$

Exercise: • $\left(V/\mathbb{F}[X_0]\{\xi_{X_0}\mid\xi\in\mathfrak{g}\}\right)^G$ naturally acts on $\mathbb{F}[Y_0]=$
$=\mathbb{F}[X_0]^G$ by derivations. This gives a map $\longrightarrow \mathcal{U}$

• This map is an isom'm of $\mathbb{F}[Y_0]$-modules.

(hint: – treat the case of $X_0=G\times Y_0$

– reduce to this case by applying the functor
$\mathbb{F}[\tilde{Y}_0]\otimes_{\mathbb{F}[Y_0]}\bullet$ to the homom'm. $\tilde{Y}_0\to Y_0$ is surjective & etale

Our functor is exact & if a morphism goes to isomorphism, then it's an isom'm itself.)

$$B_0 = \mathbb{F}[Y_0] = \mathbb{F}[X_0]^G \longrightarrow (\mathcal{D}(X_0)/\mathcal{D}(X_0)\{\xi_{X_0}\})^G$$
- from $\mathbb{F}[X_0] \hookrightarrow \mathcal{D}(X_0)$

$$U = (V/\mathbb{F}[X_0]\{\xi_{X_0}\})^G \longrightarrow (\mathcal{D}(X_0)/\mathcal{D}(X_0)\{\xi_{X_0}\})^G$$
- from $V \hookrightarrow \mathcal{D}(X_0)$

Exercise: Maps $B_0, U \to (\mathcal{D}(X_0)/\mathcal{D}(X_0)\{\xi_{X_0}\})^G$ extend to homom'm from $\mathcal{D}(Y_0)$.

Exercise: This homom'm is an isomorphism (first prove for $X_0 = G \times Y_0$, then reduce using etale base change). $\square$

Rem: • Isomorphisms $\mathcal{D}(Y_0) \xrightarrow{\sim} \mathcal{D}(X_0)///_0 G$ are natural so glue together. Let $\pi: X_0 \to Y_0$ principal G-bundle, where Y_0 is not required to be affine. Then can do sheaf version of quantum Ham. red'n:

Can view $\mathcal{D}_{X_0}/\mathcal{D}_{X_0}\{\xi_{X_0}\}$ as a G-equiv't quasi-coh't sheaf on X_0

Set $\mathcal{D}_{X_0}///_0 G := [\pi_*(\mathcal{D}_{X_0}/\mathcal{D}_{X_0}\{\xi_{X_0}\})]^G$

If we cover $Y_0 = \bigcup_i Y_0^i$ (open affines) $\Rightarrow X_0^i := \pi^{-1}(Y_0^i) \to Y_0^i$ is princ. G-bundle & $\Gamma(Y_0^i, \mathcal{D}_{X_0}///_0 G) := \mathcal{D}(X_0^i)///_0 G$.

Prop'n $\Rightarrow \mathcal{D}_{X_0}///_0 G \xrightarrow{\sim} \mathcal{D}_{Y_0}$.

• If Y_0 is affine, then classical reduction $\mathbb{F}[T^*X_0]///_0 G \xrightarrow{\sim} \mathbb{F}[T^*Y_0]$.

2.2) Twisted diff'l operators

Y_0 smooth & affine, $\mathcal{L}$ line bundle.

$X_0 = (\text{Total space of } \mathcal{L}) \setminus Y_0$ - locally trivial principal $\mathbb{F}^\times$-bundle over Y_0. $\Rightarrow \mathcal{D}(X_0)///_0 \mathbb{F}^\times \xrightarrow{\sim} \mathcal{D}(Y_0)$

$$\begin{array}{ccc} Lie(\mathbb{F}^\times) & \xrightarrow{\sim} & \mathbb{F} \\ \Psi & & \Psi \\ d_1(id) & \longmapsto & 1 \end{array}$$

Want to understand $\mathcal{D}(X_0)///_1 \mathbb{F}^\times = [\mathcal{D}(X_0)/\mathcal{D}(X_0)(1_{X_0}-1)]^{\mathbb{F}^\times}$.

Exercise: If $\mathcal{L}$ is trivial $\Rightarrow \mathcal{D}(X_0)///_1 \mathbb{F}^\times \simeq \mathcal{D}(Y_0)$.
$(X_0 = \mathbb{F}^\times \times Y_0)$

Def'n: $\mathcal{D}(X_0)///_1 \mathbb{F}^\times$ is called the algebra of diff'l operators in $\mathcal{L}$.

Exercise: i) Let M be an $\mathcal{A}$-module w. rational G-action s.t.

- $\mathcal{A} \otimes M \to M$ is G-equiv't.
- $\Phi(\xi)m = \xi_M m \quad \forall \xi \in \mathfrak{g},\ m \in M$.

Here M is called strongly equiv't.

Then M^G is $\mathcal{A}///_0 G$-module: $(a + \mathcal{A}\Phi(\xi))m = am \quad \forall m \in M^G$

(l.h.s is well-defined b/c $\Phi(\xi)m = \xi_M m = 0$
b/c m is G-invariant.

ii) $\chi: G \to \mathbb{F}^\times$ be char'r $\rightsquigarrow d_1\chi \in (\mathfrak{g}^*)^G$ (abuse not'n & $\chi := d_1\chi$). Then semiinvariants $M^{G,\chi} = \{m \in M \mid gm = \chi(g)m\}$ is a module $\mathcal{A}///_\chi G$.

Apply this to $\mathcal{A} = \mathcal{D}(X_0)$, $M = \mathbb{F}[X_0] \curvearrowleft \mathbb{F}^\times$

$$\mathbb{F}[X_0] = \bigoplus_{i\in\mathbb{Z}} \Gamma(X, \mathcal{L}^{\otimes i})$$

$$\circlearrowleft$$

$$\mathbb{F}^\times \text{ by } t \mapsto t^i$$

For $\chi = id \Rightarrow \mathbb{F}[X_0]^{G,\chi} = \Gamma(X, \mathcal{L}) \curvearrowleft \mathcal{D}(X_0)///_1 \mathbb{F}^\times$.

Not'n: $\mathcal{D}(X_0)///_1 \mathbb{F}^\times =: \mathcal{D}(X, \mathcal{L})$.

2.3) G simple alg. gr'p $/\mathbb{F} = \mathbb{C}$, $X_0 = \mathfrak{g}$.

$G \curvearrowright \mathfrak{g}$ is not free.

Want: $\mathcal{D}(\mathfrak{g})///_0 G$.

Let $\mathfrak{h} \subset \mathfrak{g}$ be Cartan, $W \curvearrowright \mathfrak{h}$ Weyl group $\rightsquigarrow W \curvearrowright \mathcal{D}(\mathfrak{h})$ $\rightsquigarrow \mathcal{D}(\mathfrak{h})^W$.

Thm i, Harish-Chandra: have filt. alg. homom.

$$\mathcal{D}(\mathfrak{g})///_0 G \longrightarrow \mathcal{D}(\mathfrak{h})^W.$$

ii, Levasseur-Stafford: this an isomorphism.

Chevalley restriction thm: Restricting from $\mathfrak{g} \to \mathfrak{h}$ defines isom'm $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W$.

$$\mathcal{D}(\mathfrak{h})^W \curvearrowright \mathbb{C}[\mathfrak{h}]^W \text{ (faithful)}$$

Last exercise: $\mathcal{D}(\mathfrak{g})///_0 G \curvearrowright \mathbb{C}[\mathfrak{g}]^G$.

If $\mathcal{D}(\mathfrak{g})///_0 G$ acts by operators from $\mathcal{D}(\mathfrak{h})^W$, then we have our homomorphism. But this is false:

Example: Pick $(\cdot,\cdot)$ invariant orthog. form on $\mathfrak{g}$

$\rightsquigarrow \Delta_{\mathfrak{g}} \in S(\mathfrak{g})^G \subset \mathcal{D}(\mathfrak{g})^G$ (as operators w. const. coeff.)

$(\cdot,\cdot)|_{\mathfrak{h}} \rightsquigarrow \Delta_{\mathfrak{h}} \in S(\mathfrak{h})^W \subset \mathcal{D}(\mathfrak{h})^W$.

Claim/exercise: on $\mathbb{C}[\mathfrak{g}]^G$, $\Delta_{\mathfrak{g}}$ acts as $\Delta_{\mathfrak{h}} + \sum_{\alpha>0} \frac{2\partial_{\alpha^\vee}}{\alpha}$

Let $\delta := \prod_{\alpha>0} \alpha \in \mathbb{C}[\mathfrak{h}]^{W,sgn}$.

Claim/exercise: $\Delta_{\mathfrak{h}} = \delta^{-1} \circ \Delta_{\mathfrak{g}} \circ \delta$ (i.e. $\Delta_{\mathfrak{h}}(F) = \delta^{-1}\Delta_{\mathfrak{g}}(\delta F)$)

Then homomorphism is $\underset{\substack{\cap \\ \mathcal{D}(\mathfrak{g})/\!/\!/_0 G \\ \text{acting on } \mathbb{C}[\mathfrak{g}]^G}}{\alpha} \mapsto \underbrace{\delta^{-1} \circ \alpha \circ \delta}_{\text{el't of } \mathcal{D}(\mathfrak{h})^W \text{ in } \mathbb{C}[\mathfrak{h}]^W}$.

On Poisson level, still have $\mathbb{F}[T^*\mathfrak{g}]/\!/\!/_0 G$ to $\mathbb{F}[T^*\mathfrak{h}]^W$ (by restricting to $\mathfrak{h}\times\mathfrak{h} \subset \mathfrak{g}\times\mathfrak{g}$). that identifies

$(\mathbb{F}[T^*\mathfrak{g}]/\!/\!/_0 G)/\text{radical} \xrightarrow{\sim} \mathbb{F}[T^*\mathfrak{h}]^W$.

When is the radical 0?

Known for: $\mathfrak{g} = \mathfrak{gl}_n$ (Etingof-Ginzburg)

$\mathfrak{g} = \mathfrak{sp}_{2n}$ (T.-H. Chen-Ngo; Losev).