Quantizations in char p, Lecture 10
RCA's & lifting P to characteristic 0

1) Recap: $X^*_F := \text{Hilb}_n (k^2)$ We have: a $G_m$-equivariant vector bundle $P^*_F$ on $X^*_F$, $\text{End}(P^*_F) = k[V] \# S_n$ (as graded $k[V]^{S_n}$-algebras)
$\text{Ext}^i(P^*_F, P^*_F) = 0 \neq 0$.

Need: to lift $P$ to char 0. Steps to do this:

(i) $P^*_F$ is defined over $F_q \sim P^*_F$ on $X^*_F$.
(ii) $S :=$ alg. extension of $k^*$ w. residue field $F_q$,
$\hat{S} := \ker [S \rightarrow F_q]$ completion $\hat{S}$:
$X^*_S :=$ formal neigh. of $X^*_F$ in $X^*_S$, formal scheme.

Can deform $P^*_F$ to a $G_m$-equivariant vector bundle $\hat{P}^*_S$ on $X^*_S$ (by $\text{Ext}^i(P^*_F, P^*_F) = 0, i=1,2$). Thx to $G_m$-equiv. can extend $\hat{P}^*_S$ to vector bundle $P^*_S$ on $X^*_S$.

(iii) Use $\hat{S} \rightarrow S \rightarrow P^*_F$.

Want to show: $\text{End}(\hat{P}^*_S) \sim$ the $\text{M-adic}$ completion of $S[V] \# S_n$.
We know $\text{End}(\hat{P}^*_S) \otimes_S F_q \sim F_q[V] \# S_n$.

Rem: In def'n of Procesi bundle: two normalization conditions
(achieved by twisting w. line bundle / passing to dual) & more
equivariance: w.r.t. $T = (C^*)^2$, so far only have equiv. w.r.t.
diagonal $C^*$. To recover $T$-equivariance: to track the construction
(Lec 9) or use classification (uniqueness).
1) Rational Cherednik algebras

Slight change of setting: Before $\mathfrak{g} = \mathbb{C}^n$, Cartan in $\mathfrak{g}_{\mathfrak{h}}$
Now $\mathfrak{g} = \text{red}\ln \text{rep}'n$ of $S_n$, now irreducible ($\odot$), Cartan in $S_n$.
$V = \mathfrak{h} \oplus \mathfrak{h}^*$, $R_i = 2 \mathfrak{h} \oplus \mathbb{C} \sim \rho_i: T^* R \to \mathfrak{g}_{\mathfrak{h}} \sim X, Y$
Old $X = \text{new } X \times \mathbb{A}^2$. Have Procesi bundle on new $X, P$.
Old $P \sim \text{new } P \otimes \mathcal{O}_{\mathbb{A}^2}$.

$H_0 = \mathbb{F}[V] \# S_n$. I'm interested in graded deformations of $H_0$.
graded algebras $H_p$ over $\mathbb{F}[p]$, $p$ is finite dim'l vector space s.t.
• $H_p$ is free over $\mathbb{F}[p]$
• $p^* \in \mathbb{F}[p]$ has degree 2
• $H_p / (p) \sim H_0$ (as graded algebras)

Turns out $\exists$ universal such deformation $H_{tc}$ over $\mathbb{F}[t, c]$
"Universal" means: $\exists ! p \mapsto \text{Span}_\mathbb{F}(t, c)$ (linear map) & graded algebra iso $H_p \sim \mathbb{F}[p] \otimes \mathbb{F}[t, c] H_{tc}$ of deform's of $H_0$.

$H_{tc} = T(V) \# S_n [t, c] / \left( \begin{array}{l}
[y, y'] = [x, x'] = 0, \ y, y' \in \mathfrak{h},\ x, x' \in \mathfrak{h}^* \\
[y, x] = t < y, x > - c \sum_{t \in \mathbb{C}} (x-y)(y-y)(t) \end{array} \right)$

universal RCA

How to see universal property: computation of suitable graded components of $HH^i(\mathbb{F}[V] \# S_n)$, $i = 1, 2, 3$

$\zeta, \chi \in \mathbb{F} \mapsto H_{tc} / (t-\zeta, c-\chi) \sim eH_{tc} e$,
\( e = \frac{1}{15_n} \sum_{\epsilon \in S_n} \epsilon \in \mathbb{F} S_n \).
**Fact 1** (Etingof-Ginzburg): TFAE:

(i) $e H_c$ is commutative.

(ii) $e = 0$.

**Notation** $H_c := H_c / (t)$.

2) Deformation of Hilbert scheme & geometric meaning of $H_c$.

$z :=$ center of $g = g^e$, scalar matrices.

$z \in \mathfrak{z} \cap o \mathfrak{z} \sim p^{-1}(z) \subset \mathbb{T}^* R$

**Fact 2**: all $G$-orbits in $p^{-1}(z)$ are free, so closed.

$a$ affine variety $p^{-1}(z) / G$, Calogero-Moser space.

Universal reductions: $X_z := p^{-1}(z) / H^0 G, \ Y_z := p^{-1}(z) / \mathfrak{g} G$

$X_z \longrightarrow Y_z \longrightarrow z$

The fiber of $X_z$ over 0 is $X$, over $z \neq 0$, it's $p^{-1}(z) / G$.

Try $X_z, Y_z$ containing contracting torus.

Thm (Etingof-Ginzburg): There exists a graded algebra isomorphism

$F[\mathfrak{z}] \hat{\longrightarrow} e H_c$ (linear $H_c \rightarrow \mathfrak{g}^*$) deforms $F[\mathfrak{z}] \hat{\longrightarrow} e H_c$.

2: To extend this to isomorphism in target $H_c$. 

Since $\text{Ext}^i(P,P)=0$ for $i=1,2$, can uniquely deform it to formal nghd of $X$ in $X_I$, then use $G_m$-equiv to extend to $X_I$. Denote the result by $P_I$. Notice:

- $\text{Ext}^i(P_I,P_I)=0 \forall i \geq 0$
- $\text{End}(P_I)/(\mathfrak{m}^*) \sim \text{End}(P)=H_0$

**Thm (I.C.)** $\text{End}(P_I) \sim H_0$, an iso of graded $\mathbb{F}[Y_I] \sim eH_0e$ -algebras and of deformations of $H_0$.

**Sketch of proof:** $\text{End}(P_I)$ is deformation of $H_0$. Use universal property of $H_0 \sim Z \rightarrow \text{Span}_{\mathbb{F}}(t,e)$ s.t.

$$\text{End}(P_I) \sim [\mathbb{F}[Z] \otimes_{\mathbb{F}[t,e]} H_e]$$

$$\mathbb{F}[Y_I] \sim e \text{End}(P_I)e \sim [\mathbb{F}[Z] \otimes_{\mathbb{F}[t,e]} eH_0e$$

commutative $\Rightarrow \text{im } Z \subset [\mathbb{F}e$ by Fact 1.

Also $[\mathbb{F}[Y_I]$ is nontrivial deformation of $[\mathbb{F}[Y] \Rightarrow \text{the map}$

$Z \rightarrow [\mathbb{F}e$ is nonzero.

3) Lifting to char 0:

- $[\Gamma(P_I)]=[eP_I \sim O_X \otimes \mathbb{C}$ of uniqueness of deformation] $= \Gamma(P_I \otimes P_I^*)c$
- $\text{End}(P_I)e=H_0e$, an iso of $[\mathbb{F}[Y_I]=eH_0e$ -modules

- $X_I \rightarrow Y_I$ is an isomorphism outside of codim 3(!!!) locus in $Y_I$
b/c it is an isomorphism over $\mathbb{F}_q$ & $X \to Y$ is iso outside $\text{codim} \, \mathbb{Z}$

locus in $Y = Y \setminus (V^o/S_n)$.

Let $X^o_3 \hookrightarrow X_3, Y_3$ is locus of isomorphism, $P^o_3 = P_3 \mid X^o_3$.

Exercise: $\text{End} \,(P_3) \xrightarrow{\sim} \text{End} \,(P^o_3)$.

$P_3$ is defined over $\mathbb{F}_q$, $X_3/\mathbb{F}_q \hookrightarrow$ formal neigh & $\hat{X}_3/\mathbb{F}_q \hookrightarrow$ formal neigh of $X_3/\mathbb{F}_q \hookrightarrow X_3/\mathbb{F}_q$.

$\text{Ext}^i(P_3, P_3, \mathbb{F}_q) = 0, i = 1, 2 \Rightarrow$ unique deformation $P^o_3$, vector bundle on formal scheme $\hat{X}_3/\mathbb{F}_q$: $P^o_3 = P_3 \mid X^o_3$, deformation of $P^o_3$.

Notice: $\text{End} \,(P^o_3) \xrightarrow{\sim} \text{End} \,(\hat{P}_3, \mathbb{F}_q)$.

Main Lemma: $\text{End} \,(\hat{P}_3, \mathbb{F}_q) \xrightarrow{\sim} \hat{H}_{\mathbb{C}^*}, \text{Gm-equivariant isomorphism of algberas over } \hat{S}[\hat{X}_3, \mathbb{F}_q] = e \hat{H}_{\mathbb{C}^*}$.

Proof: Step 1: Claim: $\text{Ext}^n(\hat{P}_3, \hat{P}_3, \mathbb{F}_q) = 0$.

$P^o_3$ is vector bundle $\subseteq \text{H}^1(X^o_3, \text{End}(\hat{P}_3, \mathbb{F}_q))$

$\text{End}(\hat{P}_3, \mathbb{F}_q) \xrightarrow{\sim} \text{H}_{\mathbb{C}^*} \Rightarrow \text{End}(\hat{P}_3, \mathbb{F}_q) = \text{End}(\hat{P}_3, \mathbb{F}_q) \mid X^o_3 = \text{H}_{\mathbb{C}^*} \mid X^o_3$.
Subclaim: $H_{\mathfrak{c}, \mathcal{F}}$ is maximal Cohen-Macaulay (CM) module over $\mathbb{F}_q[\mathcal{F}]$.

Reason: $\mathbb{F}_q[\mathcal{F}]$ is CM ring $\Rightarrow$ (maximal) CM module over $\mathbb{F}_q[\mathcal{F}] \\ 
\Rightarrow H_0 = \mathbb{F}_q[\mathcal{F}] \cong \mathbb{F}_q[\mathcal{F}] \oplus \mathbb{F}_q^1$ is max CM module. \\
$\Rightarrow H_{\mathfrak{c}}$ is (max) CM $\mathbb{F}_q[\mathcal{F}]$-module as deformation of (max) CM module. Proves subclaim.

How does this imply the claim:

Let $Z$ be affine scheme, $Z_0$ closed subscheme:

(i) if $\mathcal{F}$ is max. CM $O_Z$ module, then $H^i_{{Z_0}}(\mathcal{F}) = 0 \forall i < \text{codim}_Z Z_0$.

(ii) $H^{i-1}(Z \setminus Z_0, \mathcal{F}) \to H^i_{{Z_0}}(\mathcal{F}) \forall i > 1$: use exact sequence \\
$\ldots \to H^{i-1}_{{Z_0}}(\mathcal{F}) \to H^i(Z, \mathcal{F}) \to H^i(Z \setminus Z_0, \mathcal{F}) \to H^{i+1}_{{Z_0}}(\mathcal{F}) \to \ldots$

Since $\text{codim}_{Z} X_{\mathfrak{c}, \mathcal{F}} \setminus X_0 = 3 \Rightarrow \text{Ext}^3(P^c_3, P^c_3) = 0$

$H^9(X_{\mathfrak{c}, \mathcal{F}}, H_{\mathfrak{c}, \mathcal{F}}) = 0$. Finishes Step 1.

max. CM by subclaim

Step 2: $\text{Ext}^2(P^c_3, P^c_3) = 0 \Rightarrow$ the deformation of $P^c_3$ to a sheaf of $\hat{X}_{\mathfrak{c}, S}$ (flat over $S$) is unique if it exists: the set of lifts from $S/\mathfrak{m}^2$ to $S/\mathfrak{m}^3$ is an affine space w. vector space $\mathfrak{m}/\mathfrak{m}^2 \otimes \text{Ext}^2(\ldots)$.

But we have a lift $\hat{X}_{\mathfrak{c}, S}$. So $\hat{P}_3^\circ \cong \hat{X}_{\mathfrak{c}, S}$, an isomorphism of $\mathbb{G}_m$-equivariant bundles on $\hat{X}_{\mathfrak{c}, S}$.

$\text{End}(\hat{P}_3^\circ) = \Gamma(\text{End}(\hat{P}_3^\circ)) = \Gamma(\text{End}(\hat{X}_{\mathfrak{c}, S})) = \Gamma(\hat{X}_{\mathfrak{c}, S}) = \hat{X}_{\mathfrak{c}, S}$. $\Box$
Cor: \( \text{End}(\hat{P}_5) \sim \hat{A}_{0,5} \)

Proof: \( \text{End}(\hat{P}_5) = \text{End}(\hat{P}_{3,5})/(\mathbb{Z}^*) \sim \text{End}(\hat{P}_{3,5}^0)/(\mathbb{Z}^*) \sim \hat{A}_{0,5}/(\mathbb{Z}) = \hat{A}_{0,5} \)

1.4) Comments:

1) In [BK], different approach is used: first handle \( \text{dim} V = 2 \), then reduce the general case to this using techniques similar to Sec 1.3.

2) Deformation over \( \mathbb{Z} \) is useful for several reasons:
   - We have similar \( \hat{P}_5 \) over \( \mathbb{C} \).
   - It allows to use uniqueness of Procesi bundle: recover \( \hat{P}_5 \mid_{x^0} \) from \( \Gamma(\hat{P}_5 \mid_{x^0}) = H_c \).
   - Next lecture: we’ll this deformation as one (of two) ingredients to establish Macdonald positivity.
   - Rational Cherednik algebras are just COOL!!!