Quantizations in char p, Lecture 10.
RCA's & lifting $\mathcal{P}$ to characteristic 0.

0) Recap: $X_{\mathbb{F}} := \mathrm{Hilb}_n(\mathbb{F}^2)$. We have: a $\mathbb{G}_m$-equivariant vector bundle $\mathcal{P}_{\mathbb{F}}$ on $X_{\mathbb{F}}$ w. $End(\mathcal{P}_{\mathbb{F}}) = \mathbb{F}[V]\# S_n$ (as graded $\mathbb{F}[V]^{S_n}$-algebras) $Ext^i(\mathcal{P}_{\mathbb{F}}, \mathcal{P}_{\mathbb{F}}) = 0 \ \forall i > 0$.

Need: to lift $\mathcal{P}$ to char 0. Steps to do this:

(i) $\mathcal{P}_{\mathbb{F}}$ is defined over $\mathbb{F}_q \rightsquigarrow \mathcal{P}_{\mathbb{F}_q}$ on $X_{\mathbb{F}_q}$

(ii) $S :=$ alg. extension of $\mathbb{Z}$ w. residue field $\mathbb{F}_q$, $\mathfrak{m} := \ker[S \twoheadrightarrow \mathbb{F}_q]$ $\rightsquigarrow$ completion $\hat{S}$; $\hat{X}_S =$ formal neigh.d of $X_{\mathbb{F}_q}$ in X_S, formal scheme. Can deform $\mathcal{P}_{\mathbb{F}_q}$ to a $\mathbb{G}_m$-equiv't vector bundle $\hat{\mathcal{P}}_S$ on $\hat{X}_S$ (b/c $Ext^i(\mathcal{P}_{\mathbb{F}_q}, \mathcal{P}_{\mathbb{F}_q}) = 0, i = 1,2$). Thx to $\mathbb{G}_m$-equiv'e can extend $\hat{\mathcal{P}}_S$ to vector bundle $\mathcal{P}_{\hat{S}}$ on $X_{\hat{S}}$.

(iii) Use $\hat{S} \hookrightarrow \mathbb{C} \rightsquigarrow \mathcal{P}_{\mathbb{C}}$.

Want to show: $End(\hat{\mathcal{P}}_S) \xrightarrow{\sim}$ the $\mathfrak{m}$-adic completion of $S[V]\# S_n$.
We know $End(\hat{\mathcal{P}}_S) \otimes_{\hat{S}} \mathbb{F}_q \xrightarrow{\sim} \mathbb{F}_q[V]\# S_n$

Rem: In def'n of Procesi bundle: two normalization conditions (achieved by twisting w. line bundle/passing to dual) & more equivariance: w.r.t. $T = (\mathbb{C}^\times)^2$, so far only have equiv'e w.r.t. diagonal $\mathbb{C}^\times$. To recover T-equivariance: to track the construction (Lec 4) or use classification (uniqueness).

1) Rational Cherednik algebras.

Slight change of setting: Before $\mathfrak{h}=\mathbb{C}^n$, Cartan in $\mathfrak{gl}_n$

Now $\mathfrak{h}$ = refl'n rep'n of S_n, now irreducible (☺), Cartan in $\mathfrak{sl}_n$.

$V=\mathfrak{h}\oplus\mathfrak{h}^*$, $R:=\mathfrak{sl}_n\oplus\mathbb{C}^n \rightsquigarrow \mu: T^*R\to\mathfrak{g}=\mathfrak{gl}_n \rightsquigarrow X, Y$

Old X = new $X\times\mathbb{A}^2$. Have Procesi bundle on new X, $\mathcal{P}$;

old $\mathcal{P}\simeq$ new $\mathcal{P}\boxtimes\mathcal{O}_{\mathbb{A}^2}$.

$H_0:=\mathbb{F}[V]\# S_n$. I'm interested in graded deformations of H_0: graded algebras $H_\mathfrak{p}$ over $\mathbb{F}[\mathfrak{p}]$, $\mathfrak{p}$ is finite dim'l vector space s.t.

- $H_\mathfrak{p}$ is free over $\mathbb{F}[\mathfrak{p}]$
- $\mathfrak{p}^*\subset\mathbb{F}[\mathfrak{p}]$ has degree 2
- $H_\mathfrak{p}/(\mathfrak{p})\xrightarrow{\sim}H_0$ (as graded algebras)

Turns out $\exists$ universal such deformation $H_{t,c}$ over $\mathbb{F}[t,c]$

"Universal" means: $\exists!\ \mathfrak{p}\longrightarrow \mathrm{Span}_\mathbb{F}(t,c)$ (linear map) & graded algebra iso $H_\mathfrak{p}\xrightarrow{\sim}\mathbb{F}[\mathfrak{p}]\otimes_{\mathbb{F}[t,c]}H_{t,c}$ of deform'ns of H_0.

$$\underbrace{H_{t,c}}_{\text{universal RCA}}=T(V)\# S_n[t,c]\Big/\left(\begin{array}{l}[y,y']=[x,x']=0,\ y,y'\in\mathfrak{h},\ x,x'\in\mathfrak{h}^*\\ [y,x]=t\langle y,x\rangle-c\sum\limits_{i<j}\underbrace{(x_i-x_j)(y_i-y_j)}_{\text{coordinates}\in\mathbb{C}}\underbrace{(ij)}_{\in S_n}\end{array}\right)$$

How to see universal property: computation of suitable graded components of $HH^i(\mathbb{F}[V]\# S_n)$, $i=1,2,3$.

$\tau,\gamma\in\mathbb{F}\rightsquigarrow H_{\tau,\gamma}=H_{t,c}/(t-\tau,c-\gamma)\rightsquigarrow eH_{\tau,\gamma}e$, $e=\frac{1}{|S_n|}\sum\limits_{\sigma\in S_n}\sigma\in\mathbb{F}S_n$.

Fact 1 (Etingof-Ginzburg): TFAE:

(i) $eH_{\tau,\gamma}e$ is commutative

(ii) $\tau=0$.

Notation $H_c := H_{t,c}/(t)$.

2) Deformation of Hilbert scheme & geometric meaning of H_c.

$\mathfrak{z} :=$ center of $\mathfrak{g}=\mathfrak{gl}_n$, scalar matrices.

$\zeta\in\mathfrak{z}\setminus\{0\} \rightsquigarrow \mu^{-1}(\zeta)\subset T^*R$

Fact 2: all G-orbits in $\mu^{-1}(\zeta)$ are free, so closed

$\rightsquigarrow$ affine variety $\mu^{-1}(\zeta)/G$, Calogero-Moser space.

Universal reductions: $X_{\mathfrak{z}} := \mu^{-1}(\mathfrak{z})/\!/^{\theta} G$, $Y_{\mathfrak{z}} := \mu^{-1}(\mathfrak{z})/\!/G \rightsquigarrow$

$X_{\mathfrak{z}} \longrightarrow Y_{\mathfrak{z}} \longrightarrow \mathfrak{z}$

The fiber of $X_{\mathfrak{z}}$ over 0 is X, over $\zeta\neq 0$, it's $\mu^{-1}(\zeta)/G$

— · — · — $Y_{\mathfrak{z}}$ — · — · — Y, — · — · — · — · — · — · —

$T \curvearrowright X_{\mathfrak{z}}, Y_{\mathfrak{z}}$ containing contracting torus.

Thm (Etingof-Ginzburg) $\exists$ graded algebra isomorphism

$\mathbb{F}[Y_{\mathfrak{z}}] \xrightarrow{\sim} eH_ce$ (linear $\mathbb{F}c\to\mathfrak{z}^*$) deforms $\mathbb{F}[Y]\xrightarrow{\sim} eH_0e$.

Q: To extend this to isomorphism w. target H_c

Since $Ext^i(\mathcal{P},\mathcal{P})=0$ for $i=1,2$, can uniquely deform it to formal nghd of X in $X_{\mathfrak{z}}$, then use $\mathbb{G}_m$-equiv'e to extend to $X_{\mathfrak{z}}$. Denote the result by $\mathcal{P}_{\mathfrak{z}}$. Notice:

- $Ext^i(\mathcal{P}_{\mathfrak{z}},\mathcal{P}_{\mathfrak{z}})=0 \ \forall\, i>0$
- $End(\mathcal{P}_{\mathfrak{z}})/(\mathfrak{z}^*) \xrightarrow{\sim} End(\mathcal{P})=H_0$.

Thm (I.L.) $End(\mathcal{P}_{\mathfrak{z}}) \xrightarrow{\sim} H_c$, an iso of graded $\mathbb{F}[Y_{\mathfrak{z}}]\simeq eH_{0,c}e$-algebras and of deformations of H_0.

Sketch of proof: $End(\mathcal{P}_{\mathfrak{z}})$ is deformation of H_0. Use universal property of $H_{t,c} \rightsquigarrow \mathfrak{z} \to Span_{\mathbb{F}}(t,c)$ s.t.

$$End(\mathcal{P}_{\mathfrak{z}}) \xrightarrow{\sim} \mathbb{F}[\mathfrak{z}]\otimes_{\mathbb{F}[t,c]} H_{t,c}$$

$$\Downarrow$$

$$\mathbb{F}[Y_{\mathfrak{z}}] = e\,End(\mathcal{P}_{\mathfrak{z}})e \xrightarrow{\sim} \mathbb{F}[\mathfrak{z}]\otimes_{\mathbb{F}[t,c]} eH_{t,c}e$$

commutative $\Rightarrow im\,\mathfrak{z}\subset \mathbb{F}c$ by Fact 1.

Also $\mathbb{F}[Y_{\mathfrak{z}}]$ is nontrivial deformation of $\mathbb{F}[Y] \Rightarrow$ the map $\mathfrak{z}\to\mathbb{F}c$ is nonzero. $\square$

3) Lifting to char 0:

- $\Gamma(\mathcal{P}_{\mathfrak{z}})=[e\mathcal{P}_{\mathfrak{z}}\simeq \mathcal{O}_{X_{\mathfrak{z}}}$ b/c of uniqueness of deformation$]=\Gamma(\mathcal{P}_{\mathfrak{z}}\otimes\mathcal{P}_{\mathfrak{z}}^*)e$ $=End(\mathcal{P}_{\mathfrak{z}})e=H_ce$, an iso of $\mathbb{F}[Y_{\mathfrak{z}}]=eH_ce$-modules

- $X_{\mathfrak{z}}\longrightarrow Y_{\mathfrak{z}}$ is an isomorphism outside of codim 3(!!!) locus in $Y_{\mathfrak{z}}$

b/c it's an isomorphism over $\mathfrak{z}\setminus\{0\}$ & $X \to Y$ is iso outside codim 2 locus in Y: $Y \setminus (V^0/S_n)$.

Let $X_{\mathfrak{z}}^0 \hookrightarrow X_{\mathfrak{z}}, Y_{\mathfrak{z}}$ is locus of isomorphism, $\mathcal{P}_{\mathfrak{z}}^0 := \mathcal{P}_{\mathfrak{z}}|_{X_{\mathfrak{z}}^0}$.

Exercise: $End(\mathcal{P}_{\mathfrak{z}}) \xrightarrow{\sim} End(\mathcal{P}_{\mathfrak{z}}^0)$.

$\mathcal{P}_{\mathfrak{z}}$ is defined over $\mathbb{F}_q$, $X_{\mathfrak{z},\mathbb{F}_q} \subset X_{\mathfrak{z},S} \rightsquigarrow$ formal neigh'd $\hat{X}_{\mathfrak{z},S}$ &
$\hat{X}_{\mathfrak{z},S}^0$ - formal neigh'd of $X_{\mathfrak{z},\mathbb{F}_q}^0 \subset X_{\mathfrak{z},S}^0$.

$Ext^i(\mathcal{P}_{\mathfrak{z},\mathbb{F}_q}, \mathcal{P}_{\mathfrak{z},\mathbb{F}_q}) = 0$, $i=1,2 \rightsquigarrow$ unique deformation $\hat{\mathcal{P}}_{\mathfrak{z},S}$, vector bundle on formal scheme $\hat{X}_{\mathfrak{z},S}$: $\hat{\mathcal{P}}_{\mathfrak{z},S}^0 := \hat{\mathcal{P}}_{\mathfrak{z},S}|_{\hat{X}_{\mathfrak{z},S}^0}$, deformation of $\mathcal{P}_{\mathfrak{z},\mathbb{F}_q}^0$.

Notice: $End(\hat{\mathcal{P}}_{\mathfrak{z},S}) \xrightarrow{\sim} End(\hat{\mathcal{P}}_{\mathfrak{z},S}^0)$.

Main lemma: $End(\hat{\mathcal{P}}_{\mathfrak{z},S}^0) \xrightarrow{\sim} \hat{H}_{c,S}$, $\mathbb{G}_m$-equivariant isomorphism of algebres over $\hat{S}[\hat{X}_{\mathfrak{z},S}] = e\hat{H}_{c,S}e$.

Proof: Step 1: Claim: $Ext^1(\mathcal{P}_{\mathfrak{z},\mathbb{F}_q}^0, \mathcal{P}_{\mathfrak{z},\mathbb{F}_q}^0) = 0$.

$\mathcal{P}_{\mathfrak{z},\mathbb{F}_q}^0$ is vector bdle $\rightsquigarrow$ ||

$$H^1(X_{\mathfrak{z},\mathbb{F}_q}^0, \mathcal{E}nd(\mathcal{P}_{\mathfrak{z},\mathbb{F}_q}^0))$$

$End(\mathcal{P}_{\mathfrak{z},\mathbb{F}_q}) \xrightarrow{\sim} H_{c,\mathbb{F}_q} \Rightarrow \mathcal{E}nd(\mathcal{P}_{\mathfrak{z},\mathbb{F}_q}^0) = \mathcal{E}nd(\mathcal{P}_{\mathfrak{z},\mathbb{F}_q})|_{X_{\mathfrak{z},\mathbb{F}_q}^0} = H_{c,\mathbb{F}_q}|_{X_{\mathfrak{z},\mathbb{F}_q}^0}$

Subclaim: $H_{c,\mathbb{F}_q}$ is maximal Cohen-Macaulay (CM) module over $\mathbb{F}_q[Y_\mathfrak{z}]$.
Reason: $\mathbb{F}[V]$ is CM ring $\Rightarrow$ (maximal) CM module over $\mathbb{F}_q[V]^{S_n} = \mathbb{F}_q[Y]$
$\Rightarrow H_0 = \mathbb{F}_q[V] \# S_n \simeq \mathbb{F}_q[V]^{\oplus |S_n|}$ is max CM module.
$\Rightarrow H_c$ is (max) CM $\mathbb{F}_q[Y_\mathfrak{z}]$-module as deformation of (max) CM module. Proves subclaim.

How does this imply the claim:
Let Z be affine scheme, Z_0 closed subscheme:
(i) if $\mathcal{F}$ is max. CM $\mathcal{O}_Z$-module, then $H^i_{Z_0}(\mathcal{F}) = 0 \ \forall i < \operatorname{codim}_Z Z_0$.
(ii) $H^{i-1}(Z \setminus Z_0, \mathcal{F}) \xrightarrow{\sim} H^i_{Z_0}(\mathcal{F}) \ \forall i > 1$: use exact sequence
$$\ldots \to H^j_{Z_0}(\mathcal{F}) \to H^j(Z, \mathcal{F}) \to H^j(Z \setminus Z_0, \mathcal{F}) \to H^{j+1}_{Z_0}(\mathcal{F}) \to \ldots$$

Since $\operatorname{codim}_{Y_{\mathfrak{z},\mathbb{F}_q}} Y_{\mathfrak{z},\mathbb{F}_q} \setminus X^o_{\mathfrak{z},\mathbb{F}_q} = 3 \Rightarrow \operatorname{Ext}^1(\mathcal{P}^o_{\mathfrak{z},\mathbb{F}_q}, \mathcal{P}^o_{\mathfrak{z},\mathbb{F}_q}) =$

$H^1(X^o_{\mathfrak{z},\mathbb{F}_q}, H_{c,\mathbb{F}_q}) = 0$. Finishes Step 1.
max. CM by Subclaim

Step 2: $\operatorname{Ext}^1(\mathcal{P}^o_{\mathfrak{z},\mathbb{F}_q}, \mathcal{P}^o_{\mathfrak{z},\mathbb{F}_q}) = 0 \Rightarrow$ the deformation of $\mathcal{P}^o_{\mathfrak{z},\mathbb{F}_q}$ to a sheaf of $\hat{X}^o_{\mathfrak{z},S}$ (flat over $\hat{S}$) is unique if it exists: the set of lifts from $S/\mathfrak{m}^k$ to $S/\mathfrak{m}^{k+1}$ is an affine space w. vector space $\mathfrak{m}^k/\mathfrak{m}^{k+1} \otimes \operatorname{Ext}^1(\ldots)$.

But we have a lift $\hat{H}^o_{c,S} e$. So $\hat{\mathcal{P}}^o_{\mathfrak{z},S} \simeq \hat{H}^o_{c,S} e$, an isom'm of $\mathbb{G}_m$-equiv't bundles on $\hat{X}^o_{\mathfrak{z},S}$.

$\operatorname{End}(\hat{\mathcal{P}}^o_{\mathfrak{z},S}) = \Gamma(\mathcal{E}nd\, \hat{\mathcal{P}}^o_{\mathfrak{z},S}) = \Gamma(\mathcal{E}nd(\hat{H}^o_{c,S} e)) = \Gamma(\hat{H}^o_{c,S}) = \hat{H}_{c,S}$. $\square$

Cor: $\mathrm{End}(\hat{\mathcal{P}}_S) \xrightarrow{\sim} \hat{H}_{0,S}$

Proof: $\mathrm{End}(\hat{\mathcal{P}}_S) = \mathrm{End}(\hat{\mathcal{P}}_{\mathfrak{z},S})/(\mathfrak{z}^*) \xrightarrow{\sim} \mathrm{End}(\hat{\mathcal{P}}^\circ_{\mathfrak{z},S})/(\mathfrak{z}^*) \xrightarrow{\sim}$
$\hat{H}_{c,S}/(c) = \hat{H}_{0,S}$ $\square$

1.4) Comments:

1) In [BK], different approach is used: first handle $\dim V=2$, then reduce the general case to this using techniques similar to Sec 1.3.

2) Deformation over $\mathfrak{z}$ is useful for several reasons:

- We have similar $\mathcal{P}_{\mathfrak{z}}$ over $\mathbb{C}$.
- it allows to use uniqueness of Procesi bundle: recover $\mathcal{P}_{\mathfrak{z}}|_{X^\circ_{\mathfrak{z}}}$ from $\Gamma(\mathcal{P}_{\mathfrak{z}}|_{X^\circ_{\mathfrak{z}}}) = H_c e$.
- next lecture: we'll this deformation as one (of two) ingredients to establish Macdonald positivity.
- Rational Cherednik algebras are just COOL!!!