Lecture 11
Macdonald positivity.

1) Statement:

$X = \mathfrak{sl}_n$-version of $Hilb_n(\mathbb{C}^2) = X \times \mathbb{C}^2$

$\mathcal{P}$ - Procesi bundle on X

$T = (\mathbb{C}^\times)^2 \curvearrowright X$ & $\mathcal{P}$ is equivariant.

$X^T \xleftrightarrow{\sim} \{\text{partitions of } n\}$

$\quad \rotatebox{90}{$\in$} \qquad\qquad\qquad \rotatebox{90}{$\in$}$

$p_\lambda \longleftrightarrow \lambda$

corresponds to monomial ideal.

$T_h = \{(t, t^{-1})\} \subset T$, $X^{T_h} = X^T$

$End(\mathcal{P}) \xrightarrow{\sim} S(\mathfrak{h} \oplus \mathfrak{h}^*) \# W$ (bigraded isomorphism), $\mathfrak{h} \subset \mathfrak{sl}_n$ is Cartan.

skyscraper at 0

Care about $\mathbb{C} \otimes^L_{S(\mathfrak{h}^*)} \mathcal{P}$ ($S(\mathfrak{h}^*) \curvearrowright \mathcal{P}$ via $S(\mathfrak{h}^*) \hookrightarrow S(\mathfrak{h} \oplus \mathfrak{h}^*) \# W = End(\mathcal{P})$, an object of $D^b(Coh\, X)$ & $\mathbb{C} \otimes^L_{S(\mathfrak{h})} \mathcal{P}$.

Thm (Macdonald positivity, geometric version)

1) $\mathbb{C} \otimes^L_{S(\mathfrak{h}^*)} \mathcal{P}$, $\mathbb{C} \otimes^L_{S(\mathfrak{h})} \mathcal{P}$ are in homol. degree 0.

Observe $S_n \curvearrowright \mathbb{C} \otimes_{S(\mathfrak{h}^*)} \mathcal{P}$, $\mathbb{C} \otimes_{S(\mathfrak{h})} \mathcal{P}$; for part'n $\lambda \leftrightarrow S_n$-irrep, let e_λ be a primitive idempotent in $\mathbb{C} S_n$ corresp to λ

$\rightsquigarrow e_\lambda(\mathbb{C} \otimes_{S(\mathfrak{h}^*)} \mathcal{P})$, $e_\lambda(\mathbb{C} \otimes_{S(\mathfrak{h})} \mathcal{P})$

2) • if $e_\lambda(\mathbb{C}\otimes_{S(\mathfrak{h}^*)}\mathcal{P})$ has nonzero fiber at p_μ, then $\mu\leq\lambda$
(in dominance order: $\mu\leq\lambda \overset{def}{\Longleftrightarrow} \sum_{i=1}^{k}\mu_i \leq \sum_{i=1}^{k}\lambda_i \ \forall k$)

transpose

• if $e_\lambda(\mathbb{C}\otimes_{S(\mathfrak{h})}\mathcal{P})$ has nonzero fiber at p_μ, then $\mu\geq\lambda^{t}$

Cor: Frobenius character of $\mathcal{P}_{p_\mu}$ is $\widetilde{H}_\mu(x;q,t)$ (Haiman's modified Macdonald polynomial)

Example: $n=2$, $X=T^*\mathbb{P}^1$, $\mathcal{P}=\mathcal{O}\oplus\mathcal{O}(1) \curvearrowleft S_2$

triv sgn

How $x\in\mathfrak{h}^*$ & $y\in\mathfrak{h}$ act on $\mathcal{O}\oplus\mathcal{O}(1)$
$\pi: T^*\mathbb{P}^1\longrightarrow\mathbb{P}^1 \rightsquigarrow \pi_*(\mathcal{O}\oplus\mathcal{O}(1))$, sheaf of modules over $\pi_*\mathcal{O}$
$\pi_*\mathcal{O} = \bigoplus_{k\geq 0}\mathcal{O}_{\mathbb{P}^1}(2k)$, $\pi_*\mathcal{O}(1)=\bigoplus_{k\geq 0}\mathcal{O}_{\mathbb{P}^1}(2k+1)$.

$\mathcal{O}_{\mathbb{P}^1}(1)\otimes\mathcal{O}_{\mathbb{P}^1}(k)\xrightarrow{\sim}\mathcal{O}_{\mathbb{P}^1}(k+1) \rightsquigarrow$ elements of $\Gamma(\mathcal{O}_{\mathbb{P}^1}(1))$ give maps $\mathcal{O}_{\mathbb{P}^1}(k)\longrightarrow\mathcal{O}_{\mathbb{P}^1}(k+1)$. The elements x,y act as T_h-eigenvectors in $\Gamma(\mathcal{O}_{\mathbb{P}^1}(1))=\mathbb{C}^2$

1) follows b/c everything is locally free.

For 2): $\mathbb{C}\otimes_{S(\mathfrak{h}^*)}\mathcal{P} = \mathcal{P}/x\mathcal{P} = \mathcal{O}\oplus\bigoplus_{k>0}\mathcal{O}_{\mathbb{P}^1}(k)/x\,\mathcal{O}_{\mathbb{P}^1}(k-1)$

Two fixed points, $[1:0]\longleftrightarrow(2)$, $[0:1]\longleftrightarrow(1^2)$
$x=$ 1st coordinate, $y=$ 2nd coordinate

$e(\mathcal{P}/x\mathcal{P}) = \mathcal{O} \oplus \bigoplus_{k>0} \mathcal{O}_{\mathbb{P}^1}(2k)/x\,\mathcal{O}_{\mathbb{P}^1}(2k-1)$ – has nonzero fiber at both fixed points

$e_{sgn}(\mathcal{P}/x\mathcal{P}) = \bigoplus_{k>0} \mathcal{O}_{\mathbb{P}^1}(2k-1)/x\,\mathcal{O}_{\mathbb{P}^1}(2k-2)$ only supported at $[0:1]$

Rem: The conditions in pt 2 (& 1) of Thm are equivalent: there's autom'm of X: $x \leftrightarrow y$, T: $(t_1, t_2) \leftrightarrow (t_2, t_1)$.

2) Contracting loci

Z variety $/\mathbb{C}$ w. $\mathbb{C}^\times \curvearrowright Z$.

Contracting locus: $Z^+ = \{z \in Z \mid \exists \underbrace{\lim_{t\to 0} tz}_{\in Z^T} \in Z\}$

$p \in Z^T \rightsquigarrow Z_p^+ = \{z \in Z \mid \lim_{t\to 0} tz = p\} \rightsquigarrow Z^+ = \bigsqcup_{p \in Z^T} Z_p^+$.

Exercise: Let Z' be another variety w. $\mathbb{C}^\times \curvearrowright Z'$ & proper $\mathbb{C}^\times$-equiv't morphism $\rho: Z' \to Z$. Then $Z'^+ = \rho^{-1}(Z^+)$

Example: $Z = (\mathfrak{h} \oplus \mathfrak{h}^*)/S_n$ $(= Y)$, $Z' = \mathfrak{h} \oplus \mathfrak{h}^*$ & $\mathbb{C}^\times$ acts as $T_\hbar$. $Z'^+ = \mathfrak{h} \Rightarrow Z^+ = \mathfrak{h}/W \subset Z$. In both cases, there's unique fixed point.

Facts: 0) if Z is smooth, then Z^T is smooth.

1) if Z is smooth & symplectic, $\mathbb{C}^\times \curvearrowright Z$ is Hamiltonian, $|Z^{\mathbb{C}^\times}| < \infty \Rightarrow \forall p \in Z^{\mathbb{C}^\times}$, then Z_p^+ is Lagrangian subvariety, isomorphic to affine space.

2) If Z is, in addition, affine then every Z_p^+ is closed

Examples: • $Z = T^*\mathbb{P}^1$, Hamiltonian $\mathbb{C}^\times$-action. Two fixed pts $[1:0]$ & $[0:1]$. $Z^+_{[0:1]} = T^*_{[0:1]}\mathbb{P}^1$, $Z^+_{[1:0]} = \underbrace{\mathbb{P}^1\setminus\{[0:1]\}}_{\text{not closed.}}$

• $Z=\{(x,y,z)\in\mathbb{C}^3 \mid y^2=xz+1\}$, $\mathbb{C}^\times \curvearrowright Z : t.(x,y,z)=(tx,y,t^{-1}z)$
2 fixed pts $(0,1,0)$, $(0,-1,0)$ w. attractors are $\{(x,1,0)\}$, $\{(x,-1,0)\}$.

Order on $Z^{\mathbb{C}^\times}$ (assumed to be finite) defined by $\mathbb{C}^\times \curvearrowright Z$.
Assume Z is smooth & sympl'c, $\mathbb{C}^\times \curvearrowright Z$ acts by Hamilt. action.
$p,p' \in Z^{\mathbb{C}^\times}$ define pre-order $p \le p' \overset{def}{\iff} p \in \overline{Z^+_{p'}}$
Then extend it to an order by transitivity.

Example: • $Z = T^*\mathbb{P}^1$: $[0:1] \le [1:0]$
• Z is affine, then the order is trivial
• For $Z = \mathrm{Hilb}_n(\mathbb{C}^2)$, $p_\lambda \le p_\mu \iff \lambda \le \mu$.

3) Proof of part 1 of Thm: flatness
based on construction of $\mathcal{P}$ via quantizations in char p:

$$\begin{tikzcd}
Fr_*\mathcal{O}_{X_{\mathbb{F}}} \arrow[r, "\text{quantize}"] & \mathcal{D} \arrow[r, "\text{compl'n}"] & \mathcal{D}^{\wedge_0} \arrow[r, "\text{splitting bundle}"] & \mathcal{E} \arrow[r, "\text{rearrange direct}", "\text{summands}"'] & \mathcal{P}_{\mathbb{F}}^{(1)\wedge_0} \arrow[d, "\text{extension using } \mathbb{G}_m\text{-equivariance}"'] \\
\mathcal{P} & \mathcal{P}_{\hat{S}} \arrow[l, "\mathbb{C}\otimes_{\hat{S}} \bullet"'] & \hat{\mathcal{P}}_S \arrow[l, "\mathbb{G}_m\text{-finite part}"'] & \mathcal{P}_{\mathbb{F}} \arrow[l, "\text{deform to char } 0"'] & \mathcal{P}_{\mathbb{F}}^{(1)} \arrow[l, "\text{untwist}"]
\end{tikzcd}$$

Claim 1: $\mathcal{P}$ is flat over $S(\mathfrak{h}^*)$ if $X_{\mathbb{F}}$ is flat over $\mathfrak{h}_{\mathbb{F}}/S_n$ (under $X_{\mathbb{F}} \longrightarrow (\mathfrak{h}_{\mathbb{F}} \oplus \mathfrak{h}^*_{\mathbb{F}})/S_n \longrightarrow \mathfrak{h}_{\mathbb{F}}/S_n$).

Proof:
Observation from commutative algebra: let M is $\mathbb{F}[x_1, \dots x_n]$, let $f_1, \dots f_n \in \mathbb{F}[x_1, \dots x_n]$ be a homogeneous regular sequence. Then M is flat over $\mathbb{F}[x_1, \dots x_n] \Leftrightarrow$ it's flat over $\mathbb{F}[f_1, \dots f_n]$.

Step 1: $f_1, \dots f_{n-1} \in S(\mathfrak{h}^*)^{S_n}$, minimal collection of homog. generators.
$\mathcal{P}$ is flat over $S(\mathfrak{h}^*) \Leftrightarrow \mathcal{P}$ is flat over $S(\mathfrak{h}^*)^{S_n}$ (by Observation)
Now $S(\mathfrak{h}^*)^{S_n} \curvearrowright \mathcal{P}$ comes from $S(\mathfrak{h}^*)^W \longrightarrow \mathbb{C}[X]$.
All steps in construction of $\mathcal{P}$ preserve flatness (e.g. $\mathcal{E}$ has structure sheaf as direct summand so if $\mathcal{D}^{\wedge_0}$ is flat over $S(\mathfrak{h}^{*(1)}_{\mathbb{F}})^{S_n}$, then $\mathcal{E}$ is also flat over $S(\mathfrak{h}^{*(1)}_{\mathbb{F}})^{S_n}$). So: if $Fr_* \mathcal{O}_{X_{\mathbb{F}}}$ is flat over $S(\mathfrak{h}^{*(1)}_{\mathbb{F}})^{S_n}$, then $\mathcal{P}$ is flat over $S(\mathfrak{h}^*)^{S_n}$, equiv. over $S(\mathfrak{h}^*)$.

Step 2: Use Observation again, to conclude that it's enough to show: $\mathcal{O}_{X_{\mathbb{F}}}$ is flat over $S(\mathfrak{h}^*_{\mathbb{F}})^{S_n} \Leftrightarrow X_{\mathbb{F}} \longrightarrow \mathfrak{h}_{\mathbb{F}}/W$ is flat. $\square$

Claim 2: $X_{\mathbb{F}} \xrightarrow{?} \mathfrak{h}_{\mathbb{F}}/S_n$ is flat.
Proof: This morphism is $\mathbb{G}_m$-equivariant. The varieties are smooth $\Rightarrow$ it's enough to check that the morphism is equidimensional (all components of all fibers have the same dim'n).

$\mathbb{G}_m$-equivariance $\rightsquigarrow$ it's enough to show $\dim \eta^{-1}(0) = \frac{1}{2}\dim X$

$\rho: X_{\mathbb{F}} \to Y_{\mathbb{F}} = (\mathfrak{h}_{\mathbb{F}} \oplus \mathfrak{h}_{\mathbb{F}}^*)/S_n$ so $\eta^{-1}(0) = \rho^{-1}(\mathfrak{h}_{\mathbb{F}}^*/S_n) =$

$= \rho^{-1}(Y_{\mathbb{F}}^+) = X_{\mathbb{F}}^+$ - union of affine spaces of $\dim = \frac{1}{2}\dim X$.

Conclude: $\dim \eta^{-1}(0) = \frac{1}{2}\dim X$ $\square$

4) Proof of 2) in Thm: Supports - using deformations.

$\mathfrak{z} := \mathfrak{z}(\mathfrak{gl}_n) = \mathbb{C} \rightsquigarrow$ deformation $X_{\mathfrak{z}}$ of X, $X_{\mathfrak{z}} = \mu^{-1}(\mathfrak{z})/\!/^{\theta} G$.

For $\gamma \in \mathfrak{z}\setminus\{0\}$, $X_\gamma = \operatorname{Spec}(eH_\gamma e)$ (H_γ is rational Cherednik algebra at $t=0$).

H_γ is filtered deformation of $S(\mathfrak{h}\oplus\mathfrak{h}^*)\# W = End(\mathcal{P})$

$\mathcal{P}$ deforms to a vector bundle $\mathcal{P}_{\mathfrak{z}}$ on $X_{\mathfrak{z}}$ w. $End(\mathcal{P}_{\mathfrak{z}}) = H_c$

$\mathcal{P}_\gamma = H_\gamma e$, an isomorphism of $eH_\gamma e = \mathbb{C}[X_\gamma]$-modules.

Point: While $\mathbb{C}\otimes_{S(\mathfrak{h}^*)}\mathcal{P}$ is hard to understand, the generic fiber $\mathbb{C}\otimes_{S(\mathfrak{h}^*)}\mathcal{P}_\gamma$ of the flat deformation $\mathbb{C}\otimes_{S(\mathfrak{h}^*)}\mathcal{P}_{\mathfrak{z}}$ is easy to understand

from 1) of Thm.

Observation: there's natural identification between X^{T_h} & $X_\gamma^{T_h}$ ($T_h \curvearrowright X_{\mathfrak{z}}$ acting trivially on $\mathfrak{z}$). For this, let's consider $X_{\mathfrak{z}}^{T_h}$

Exercise: this is the union of several copies of $\mathbb{A}^1$ each projecting isomorphically to $\mathfrak{z}$.

Bijection: $X^{T_h} \xleftrightarrow{\sim} \{\text{components of } X_{\mathfrak{z}}^{T_h}\} \xleftrightarrow{\sim} X_\gamma^{T_h}$.

Partition $\mu \rightsquigarrow$ attractor $X^+_{\gamma,\mu}$ at the fixed pt corresp. to μ, affine space of $\dim = \frac{1}{2}\dim$, closed.

Prop'n: $\operatorname{Supp}(e_\lambda(\mathbb{C}\otimes_{S(\mathfrak{h}^*)}\mathcal{P}_\gamma)) \subset X^+_{\gamma,\lambda}$.

Rem: $e_\lambda(\mathbb{C}\otimes_{S(\mathfrak{h}^*)}\mathcal{P}_\gamma) = \mathbb{C}[X^+_{\gamma,\lambda}]$.

Sketch of proof: $e_\lambda(\mathbb{C}\otimes_{S(\mathfrak{h}^*)}\mathcal{P}_\gamma) = e_\lambda(\mathbb{C}\otimes_{S(\mathfrak{h}^*)} H_\gamma e)$ as $eH_\gamma e$-module. Use Morita equiv. $eH_\gamma e$ & H_γ to exchange this to $e_\lambda(\mathbb{C}\otimes_{S(\mathfrak{h}^*)} H_\gamma) =: \Delta_\lambda$ – Verma module!

Namely, since H_γ is filt. deform'n of $S(\mathfrak{h}\oplus\mathfrak{h}^*)\# S_n$ have $H_\gamma \xleftarrow{\sim} S(\mathfrak{h}^*)\otimes\mathbb{C}S_n\otimes S(\mathfrak{h})$. As $(S(\mathfrak{h})\# S_n)^{opp}$-module have $e_\lambda(\mathbb{C}\otimes_{S(\mathfrak{h}^*)} H_\gamma) \simeq \lambda\otimes S(\mathfrak{h})$.

Observation: Δ_λ is indecomposable T_h-equivariant module & these modules are pairwise non-isomorphic. Let $\xi: X_\gamma \longrightarrow \mathfrak{h}/S_n$, natural morphism, Δ_λ is supported on $\xi^{-1}(0) = X^+_\gamma$. Since Δ_λ is indecomposable, so Δ_λ is supported on a single component $X^+_{\gamma,\lambda'}$.

Iain Gordon checked that $\lambda = \lambda'$ $\square$

Corollary: $e_\lambda(\mathbb{C}\otimes_{S(\mathfrak{h}^*)}\mathcal{P})$ is supported on $X \cap \overline{T_c X^+_{\gamma,\lambda}}$

Fact (Webster): $X\cap \overline{T_c X^+_{\gamma,\lambda}} \subset \bigcup_{\mu\leq\lambda} X^+_\mu$.

– this is what we need to prove.

Example: $n=2$, $X = T^*\mathbb{P}^1$, $X_{\mathfrak{z}} = G \times^B \mathfrak{b} = \{(x,V) \mid x \in \mathfrak{sl}_2,\ V \subset \mathbb{C}^2;$ 1-dim'l, $xV \subset V\}$, $X_{\mathfrak{z}} \to \mathfrak{z} : (x,V) \mapsto$ eigenvalue of x on V.

$T_h = T \subset SL_2$ max. torus, T_c – fiberwise dilations

$\exists \lim\limits_{t\to 0} t.(x,V) \Longleftrightarrow x$ lies in the positive Borel, $x(V_0) \subset V_0$
(for Borel fixed 1-dimensional subspace in $\mathbb{C}^2$)

$(x,V) \in X_1$ so x has eigenvalue 1 on V.

Two cases: i) $V_0 = V$.

$$X \cap \overline{T_c\{(x,V_0) \in X_1\}} = T^*_{[0:1]}\mathbb{P}^1$$

ii) $V_0 \neq V$: $V =$ 1-eigenspace for x, $V_0 = -1$-eigenspace

$$x = \begin{pmatrix} 1 & z \\ 0 & -1 \end{pmatrix}.$$

Exercise: $X \cap \overline{T_c\left\{\left(\begin{pmatrix} 1 & z \\ 0 & -1 \end{pmatrix}, V\right)\right\}} = T^*_{[0:1]}\mathbb{P}^1 \cup \mathbb{P}^1.$