Quantization commutes with reduction

Let $\mathcal{A}_\hbar$ be a formal quantization of A w. G-action & quantum comoment map $\Phi_\hbar$ as in Lecture 1. Then we have a classical comoment map $\varphi: \mathfrak{g} \to A$, $\varphi = \Phi_\hbar \bmod \hbar$. We want to have sufficient conditions for $(\mathcal{A}_\hbar/\mathcal{A}_\hbar \Phi_\hbar(\mathfrak{g}))^G$ being a formal quantization of $(A/A\varphi(\mathfrak{g}))^G$. As was discussed in the previous lecture, this is the case when the following two conditions hold:

1) $\hbar$ is not a zero divisor in $\mathcal{A}_\hbar/\mathcal{A}_\hbar\Phi_\hbar(\mathfrak{g})$
2) $(\mathcal{A}_\hbar/\mathcal{A}_\hbar \Phi_\hbar(\mathfrak{g}))^G \longrightarrow (A/A\varphi(\mathfrak{g}))^G$ is surjective.

Sufficient condition for 1):

Definition: We say that $f_1, \dots f_k \in A$ form a regular sequence if f_i is not a zero divisor in $A/(f_1, \dots f_{i-1})$ $\forall i = 1, \dots k$.

Proposition 1: Let $\xi_1, \dots \xi_n$ be a basis of $\mathfrak{g}$. Suppose that $\varphi(\xi_1), \dots \varphi(\xi_n)$ form a regular sequence in A. Then $\hbar$ is not a zero divisor in $\mathcal{A}_\hbar/\mathcal{A}_\hbar \Phi_\hbar(\mathfrak{g})$.

Proof: Let $a \in \mathcal{A}_\hbar$ satisfy $\hbar a \in \mathcal{A}_\hbar \Phi_\hbar(\mathfrak{g})$. We want to prove that $a \in \mathcal{A}_\hbar \Phi_\hbar(\mathfrak{g})$. Let $a_i \in \mathcal{A}_\hbar$ satisfy $\hbar a = \sum_{i=1}^k a_i \Phi_\hbar(\xi_i)$. Let $\underline{a}_i \in A$ be $a_i \bmod \hbar$. Then

(1) $\quad \sum_{i=1}^k \underline{a}_i \varphi(\xi_i) = 0.$

Now we use that the elements $\varphi(\xi_i)$ form a regular sequence. By Chapter 17 in Eisenbud, this implies that the higher homology of the Koszul complex for $\varphi(\xi_1), \dots \varphi(\xi_n)$ vanish.

For the 1st homology group this means the following: $\exists\ \underline{a}_{ij} \in A$ w $\underline{a}_{ii}=0$ & $\underline{a}_{ij}=-\underline{a}_{ji}$ s.t.

$$(2) \qquad \underline{a}_i = \sum_{j=1}^{n} \underline{a}_{ij}\, \varphi(\xi_j) \quad \forall\, i=1,\dots n.$$

((1) means that the element $(\underline{a}_1,\dots \underline{a}_n)$ is a cycle & (2) means it's a boundary).

Lift $\underline{a}_{ij} \in A$ to $a_{ij} \in \mathcal{A}_\hbar$ w. $a_{ii}=0$, $a_{ji}=-a_{ij}$.

Set $a_i' = \sum_{j=1}^{n} a_{ij}\, \varphi_\hbar(\xi_j)$. Note that, by the construction, $a_i' \bmod \hbar = \underline{a}_i = a_i \bmod \hbar$ so $a_i' = a_i + \hbar b_i$ for some $b_i \in \mathcal{A}_\hbar$.

So

$$\hbar a = \sum_{i=1}^{n} a_i \varphi_\hbar(\xi_i) = \sum_{i=1}^{n} a_i' \varphi_\hbar(\xi_i) + \hbar \sum_{i=1}^{n} b_i \varphi_\hbar(\xi_i)$$

$$\sum_{i=1}^{n} a_i' \varphi_\hbar(\xi_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\, \varphi_\hbar(\xi_j)\varphi_\hbar(\xi_i) = [a_{ij}=-a_{ji},\ a_{ii}=0]$$

$$= \sum_{i<j} a_{ij}\, [\varphi_\hbar(\xi_j), \varphi_\hbar(\xi_i)] = \sum_{i<j} \hbar a_{ij}\, \varphi_\hbar([\xi_j,\xi_i])$$

So $a = \sum_{i<j} a_{ij}\, \varphi_\hbar([\xi_j,\xi_i]) + \sum_i b_i \varphi_\hbar(\xi_i) \in \mathcal{A}_\hbar \varphi_\hbar(\mathfrak{g})$. $\square$

Remark: The regular sequence condition fails for $\varphi: \mathfrak{g} \to \mathbb{C}[T^*\mathfrak{g}]$.

Sufficient condition for 2): Note that 2) is satisfied when G is linearly reductive (reductive, when char $\mathbb{F}=0$ & extension of a torus $(\mathbb{F}^\times)^n$ by a finite group of order coprime to p for char $\mathbb{F}=p>0$).

Suppose from now on that X is an affine scheme (of finite type).

Then to give the classical comoment map $\varphi: \mathfrak{g} \to A := \mathbb{F}[X]$ is the same thing as to give the moment map $\mu: X \to \mathfrak{g}^*$.

Note that $A/A\varphi(\mathfrak{g})$ is nothing else but $\mathbb{F}[\mu^{-1}(0)]$, where we write $\mu^{-1}(0)$ for the scheme theoretic fiber.

Proposition 2: Suppose the G-action on $\mu^{-1}(0)$ is free & $\mu^{-1}(0)$ is a principal G-bundle over an affine scheme Y. Then

(a) $\hbar$ is not a zero divisor in $\mathcal{M}_\hbar := \mathcal{A}_\hbar / \mathcal{A}_\hbar \varphi_\hbar(\mathfrak{g})$.

(b) $\mathcal{M}_\hbar^G \twoheadrightarrow (\mathcal{M}_\hbar / \hbar \mathcal{M}_\hbar)^G$.

Sketch of proof: (a) Let $\mathcal{P}$ denote the Poisson bivector on X. The comoment map condition $\{\varphi(\xi), \cdot\} = \xi_M$ is equivalent to $\mathcal{P}(d\varphi(\xi), \cdot) = \xi_M$. Since $G \curvearrowright \mu^{-1}(0)$ freely, we have $\forall x \in \mu^{-1}(0)$ $\Rightarrow$ the vectors $(\xi_{i,M})_x$ are linearly independent. Therefore, the covectors $d_x(\varphi(\xi_i))$ are linearly independent as well. Hence μ is a submersion in x, $\forall x \in \mu^{-1}(0)$. This implies that $\varphi(\xi_1), \dots \varphi(\xi_n)$ form a regular sequence. Now (a) follows from Proposition 1.

(b): will follow if we show that

$$\forall n \quad (\mathcal{M}_\hbar / \hbar^{n+1} \mathcal{M}_\hbar)^G \twoheadrightarrow (\mathcal{M}_\hbar / \hbar^n \mathcal{M}_\hbar)^G.$$

This, in turn, will follow if we show that $Ext^1_G(\mathrm{triv}, \mathbb{F}[\mu^{-1}(0)])$

Let an etale cover $\tilde{Y} \to Y$ trivialize $\mu^{-1}(0) \xrightarrow[/G]{} Y$. As was mentioned in Lecture 1, the functor $\mathbb{F}[\tilde{Y}] \otimes_{\mathbb{F}[Y]} \bullet$ is exact & sends nonzero objects to nonzero objects. So

$$Ext^1_G(\mathrm{triv}, \mathbb{F}[\mu^{-1}(0)]) = 0 \Leftarrow 0 = \mathbb{F}[\tilde{Y}] \otimes_{\mathbb{F}[Y]} Ext^1_G(\mathrm{triv}, \mathbb{F}[\mu^{-1}(0)])$$

$$= Ext^1_G(\mathrm{triv}, \mathbb{F}[\tilde{Y}] \otimes_{\mathbb{F}[Y]} \mathbb{F}[\mu^{-1}(0)] = \mathbb{F}[\tilde{Y}] \otimes \mathbb{F}[G]) =$$

$= \mathbb{F}[\tilde{Y}] \otimes \mathrm{Ext}^1_G(\mathrm{triv}, \mathbb{F}[G])$. But $\mathbb{F}[G]$ is an injective object in the category of rational reps of $\mathbb{F}[G]$: $\mathrm{Hom}_G(V, \mathbb{F}[G]) = V^*$. So $\mathrm{Ext}^1_G(\mathrm{triv}, \mathbb{F}[G])$ vanishes finishing the proof. $\square$