Lec 2: Differential operators in char p

1. Weyl algebra
2. Differential operators.

Reference: [BMR]

1) Setting: p is prime (for applications $p \gg 0$), $\mathbb{F}=\overline{\mathbb{F}}$, $\operatorname{char}\mathbb{F}=p$.

1.1) The center:

$\mathcal{A}=\mathbb{F}\langle x,\partial\rangle/(\partial x - x\partial = 1) \curvearrowright \mathbb{F}[x]$ by diff'l operators.

If $\operatorname{char}\mathbb{F}=0$: $\mathcal{A}$ is simple, $\mathbb{F}[x]$ is simple module (& faithful) - exercise.

For $\operatorname{char}\mathbb{F}=p$: $\mathbb{F}[x]$ is neither

- faithful: $\partial^p x^n = n(n-1)\dots(n-p+1)x^{n-p}=0$
- simple: $(x^p)\subset \mathbb{F}[x]$ is a submodule.

$\mathcal{A}$ has "big center": $x^p, \partial^p \in \mathcal{A}$ are central.

Central reduction: $\alpha,\beta\in\mathbb{F} \rightsquigarrow \mathcal{A}_{\alpha,\beta}:=\mathcal{A}/(x^p-\alpha, \partial^p-\beta)$.

Proposition: i) x^p, ∂^p generate a subalgebra isomorphic to $\mathbb{F}[x^p,\partial^p]$

ii) $\mathbb{F}[x^p,\partial^p]$ = center of $\mathcal{A}$.

iii) $\mathcal{A}$ is a free rk p^2 module over $\mathbb{F}[x^p,\partial^p]$

iv) $\mathcal{A}_{\alpha,\beta}\simeq \operatorname{Mat}_p(\mathbb{F})$ $\forall \alpha,\beta\in\mathbb{F}$.

Proof:

$\mathcal{A}$ has basis of $x^i\partial^j = [i=i_0+pi_1, j=j_0+pj_1, i_0,j_0\in\{0,\dots,p-1\}]$
$= x^{i_0}\partial^{j_0}(x^p)^{i_1}(\partial^p)^{j_1} \Rightarrow$ iii) & i).

Proof of iv):

Special case: $\alpha=\beta=0$: $\mathcal{A}_{0,0}=\mathcal{A}/(x^p,\partial^p)$. Consider $\mathcal{A}$-module

$\mathbb{F}[x]/(x^p) \curvearrowleft \mathcal{A}$ factors through $\mathcal{A}_{0,0}$

Exercise: $\mathbb{F}[x]/(x^p)$ is a simple $\mathcal{A}_{0,0}$-module.

$\Rightarrow \mathcal{A}_{0,0} \twoheadrightarrow \mathrm{End}_{\mathbb{F}}(\mathbb{F}[x]/(x^p))$

have $\dim = p^2$ so get an isomorphism.

General case - α, β are arbitrary: for $a, b \in \mathbb{F}$, then $x \mapsto x+a$, $\partial \mapsto \partial + b$ extends to an autom'm of $\mathcal{A}$; $x^p \mapsto (x+a)^p = x^p + a^p$, $\partial^p \mapsto \partial^p + b^p$; take $a = \sqrt[p]{\alpha}$, $b = \sqrt[p]{\beta}$ $\rightsquigarrow$ $\mathcal{A}_{0,0} \xrightarrow{\sim} \mathcal{A}_{\alpha,\beta}$. $\square$ of iv).

Proof of ii): if $D \in$ center of $\mathcal{A}$, then it projects to (center of $\mathcal{A}_{\alpha,\beta}$) $= \mathbb{F}$ $\forall \alpha, \beta \in \mathbb{F}$.

Exercise: use that $\mathbb{F}[x^p, \partial^p]$ is normal to deduce that $\mathbb{F}[x^p, \partial^p] =$ center of $\mathcal{A}$ $\square$

Rem: Direct analog of Proposition holds for

$$D(\mathbb{A}^n) = \mathbb{F}\langle x_1, \ldots x_n, \partial_1, \ldots \partial_n \rangle / ([\partial_i, x_j] = \delta_{ij})$$

1.2) Azumaya algebras.

Let X be scheme, $\mathcal{R}$ is a coherent sheaf of $\mathcal{O}_X$-algebras. Left & right actions of $\mathcal{R}$ on itself $\rightsquigarrow$ alg. homom'm

(1) $$\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{R}^{opp} \longrightarrow \mathrm{End}_{\mathcal{O}_X}(\mathcal{R}).$$

Definition: $\mathcal{R}$ is Azumaya if

(i) $\mathcal{R}$ is a vector bundle

(ii) (1) is isomorphism.

Remark: If X is finite type scheme over $\mathbb{F}(=\overline{\mathbb{F}})$, then (ii) $\Leftrightarrow$ (ii'): $\forall x\in X \Rightarrow$ the fiber R_x is a matrix algebra.

Example 0) V is a vector bundle on $X \rightsquigarrow \mathcal{R}=\mathcal{E}nd_{\mathcal{O}_X}(V)$
Such Azumaya algebras are called split.
1) char $\mathbb{F}=p$ then $\mathcal{A}$ is an Azumaya algebra over $\mathbb{F}[x^p, \partial^p]$.

Problem: In example 1), Azumaya algebra is <u>not</u> split.

Let X be of finite type over $\mathbb{F}=\overline{\mathbb{F}}$, $x\in X \rightsquigarrow$ completion $\hat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$, complete local ring $\rightsquigarrow \hat{\mathcal{R}}_x = \hat{\mathcal{O}}_{X,x}\otimes_{\mathcal{O}_X}\mathcal{R}$.

Lemma: $\hat{\mathcal{R}}_x$ is a matrix algebra over $\hat{\mathcal{O}}_{X,x}$.
Proof: R_x is matrix algebra $\Leftrightarrow \exists$ idempotent $e\in R_x \mid$ $R_x \xrightarrow{\sim} End_{\mathbb{F}}(R_x e)$. By lifting of idempotents, $\exists$ idemp't $\hat{e}\in\hat{R}_x$ specializing to $e \Rightarrow \hat{R}_x \xrightarrow{\sim} End_{\hat{\mathcal{O}}_{X,x}}(\hat{R}_x\hat{e})$; $\hat{R}_x$ is projective $\Rightarrow \hat{R}_x\hat{e}$ is $\Rightarrow \hat{R}_x\hat{e}$ is free $\hat{\mathcal{O}}_{X,x}$-module $\square$

2) Differential operators.

Setting: $\mathbb{F}=\overline{\mathbb{F}}$, char $\mathbb{F}=p$, X_0 is smooth affine variety over $\mathbb{F}$ $\rightsquigarrow A_0=\mathbb{F}[X_0]$, $V=Vect(X_0) \rightsquigarrow \mathcal{D}(X_0)$ generated by A_0, V w. suitable relations.

Goal: generalize results of Sec 1 to $\mathcal{D}(X_0)$

2.1) Central elements.

Exercise: $\forall f \in A_0 \Rightarrow f^p$ commutes with V inside $\mathcal{D}(X_0)$ so $f^p \in Z(\mathcal{D}(X_0))$ (the center).

How about "contributions" of V to $Z(\mathcal{D}(X_0))$

Exercise: Let $\partial \in Der(A_0)\ (\hookrightarrow End_{\mathbb{F}}(A_0))$. Then $\partial^p \in End_{\mathbb{F}}(A_0)$ is a derivation

Notation: "restricted" pth power from Exercise will be denoted by $\partial^{[p]}$.

$\partial \in V \rightsquigarrow \partial^p \in \mathcal{D}(X_0)$

Proposition: $\partial^p - \partial^{[p]} \in Z(\mathcal{D}(X_0))$.

Exercise: $X_0 = \mathbb{A}^1$. Show $(x\partial)^{[p]} = x\partial$.

Remark: If $X = \mathbb{A}^n$, $\partial = \partial_i \Rightarrow \partial_i^{[p]} = 0$

Exercise: Let B be an assoc've $\mathbb{F}$-algebra, $b_1, b_2 \in B$. Then $[b_1^p, b_2] = ad(b_1)^p. b_2$.

Proof of Prop'n: $\partial^p - \partial^{[p]}$ is central $\Leftrightarrow \begin{cases} [\partial^p, f] = [\partial^{[p]}, f] & \forall f \in A_0 \\ [\partial^p, \partial'] = [\partial^p, \partial'] & \forall \partial' \in V \end{cases}$

- $[\partial^p, f] = [\text{apply Exer. to } B = \mathcal{D}(X_0)] = ad(\partial)^p f = \partial^p . f$
- $[\partial^{[p]}, f] = \partial^{[p]} f = \partial^p . f$

$[\partial^{[p]}, \partial'] = $ [apply Exer. to $B = \mathrm{End}_{\mathbb{F}}(A_0)$] $= \mathrm{ad}(\partial)^p \partial'$
$= [\partial^p, \partial']$. $\square$

2.2) Structure of the center

Q: What algebra do $f^p, \partial^p - \partial^{[p]} \in \mathcal{D}(X_0)$ generate?

Def'n: Let U be an $\mathbb{F}$-vector space. Define the Frobenius twist $U^{(1)}$ as follows:

- $U^{(1)} :=$ the same abelian group as U.
- $a \cdot^{(1)} u := a^{1/p} \cdot u \quad \forall a \in \mathbb{F}, u \in U$.

Similarly can twist associative alg's, Lie algebras etc.

Proposition: The maps $f \mapsto f^p : A_0 \to Z(\mathcal{D}(X_0))$ & $\partial \mapsto \partial^p - \partial^{[p]} : V \longrightarrow Z(\mathcal{D}(X_0))$ extend to $\mathbb{F}$-algebra homomorphism $\mathbb{F}[T^* X_0]^{(1)} \longrightarrow Z(\mathcal{D}(X_0))$, which is injective.

Lemma: $\mathcal{D}(X_0)_{\leq p-1}$ act faithfully on $\mathbb{F}[X_0]$.

Sketch of proof (of Lemma):

The claim is local. So we can assume $\exists$ etale morphism $\psi : X_0 \longrightarrow \mathbb{A}^n \rightsquigarrow x_1, \ldots x_n \in \mathbb{F}[X_0]$, $\partial_1, \ldots \partial_n \in V$ - pullbacks of eponymous elements under ψ.

$\mathcal{D}(X_0)_{\leq p-1}$ is a free left $\mathbb{F}[X_0]$-module w. basis $\partial^{\underline{i}} := \partial_1^{i_1} \ldots \partial_n^{i_n}$ w. $i_1 + \ldots + i_n \leq p-1$.

Exercise: Look at how $\sum_i f_i \partial^i$ acts on monomials x^i to conclude that $\mathcal{D}(X_0)_{\leq p-1}$ acts faithfully on $\mathbb{F}[X_0]$.

$\square$ of Lemma.

Proof of Proposition: • extension to homomorphism:

$\mathbb{F}[T^*X_0] = S_{A_0}(V)$. So we need to show that

$\eta: V \to \mathcal{Z}(\mathcal{D}(X_0))$ satisfies: $\begin{cases} \eta(f\partial) = f^p \eta(\partial) & \text{(i)} \\ \eta(\partial_1 + \partial_2) = \eta(\partial_1) + \eta(\partial_2) & \text{(ii)} \end{cases}$

$\forall f \in A_0$, $\partial, \partial_1, \partial_2 \in V$.

Proof of (i): Note $\forall \partial \in V \Rightarrow \partial^p f = \partial^{[p]} f \quad \forall f \in A_0$

So $\eta(f\partial)$ & $f^p \eta(\partial)$ act by 0 on $\mathbb{F}[X_0]$.

$\eta(f\partial) - f^p\eta(\partial) = (f\partial)^p - f^p \partial^p +$ lower order terms $\in \mathcal{D}(X_0)_{\leq p-1}$ & acts by 0 on $\mathbb{F}[X_0]$. By Lemma,

$\eta(f\partial) = f^p \eta(\partial)$

Proof of (ii): exercise

Proof of injectivity of $\mathbb{F}[T^*X_0]^{(1)} \longrightarrow \mathcal{Z}(\mathcal{D}(X_0))$

is graded

Rescale the grading on $\mathbb{F}[T^*X_0]^{(1)}$ so that V lives in deg p

Then $\mathbb{F}[T^*X_0]^{(1)} \longrightarrow \mathcal{D}(X_0)$ is a homomorphism of filt. algebras. Its associated graded is

$\mathbb{F}[T^*X_0]^{(1)} \longrightarrow \mathbb{F}[T^*X_0]$, $F \mapsto F^p$,

is injective b/c $\mathbb{F}[T^*X_0]^{(1)}$ has no nilpotent elements $\square$

Definition: Let X be a scheme over $\mathbb{F}$. Then its Frobenius twist $X^{(1)}$ is the same scheme over $\mathbb{Z}$ but with $\mathbb{F}$-linear structure twisted by $a \mapsto a^{1/p}$.

If $X = Spec(B)$, then $X^{(1)} = Spec(B^{(1)})$.

Previous prop'n: $\mathcal{D}(X_0)$ is a q-coherent sheaf of algebras on $(T^*X_0)^{(1)}$.

Thm: $\mathbb{F}[T^*X_0]^{(1)} \xrightarrow{\sim} Z(\mathcal{D}(X_0))$ & $\mathcal{D}(X_0)$ is Azumaya algebra over $(T^*X_0)^{(1)}$.

Sketch of proof: The question is local in X_0, so can assume $\exists$ etale $X_0 \to \mathbb{A}^n \rightsquigarrow$ etale $T^*X_0 \to T^*\mathbb{A}^n \rightsquigarrow$ $\varphi^{(1)}: (T^*X_0)^{(1)} \to (T^*\mathbb{A}^n)^{(1)}$.

Exercise: $\mathcal{D}(X_0) \cong_{\mathcal{O}_{(T^*X_0)^{(1)}}} (\varphi^{(1)})^* \mathcal{D}(\mathbb{A}^n)$.

This reduces the proof to case of $\mathbb{A}^n$ - Sec 1. $\square$

2.3) Concluding remarks:

0) All constr'ns in Sec 2.1 & 2.2 glue nicely. In particular, for any smooth X_0, $\mathcal{D}_{X_0}$ can be viewed as Azumaya algebra on $(T^*X_0)^{(1)}$.

1) On the Poisson level have smth similar:

Exercise: • If A is a Poisson $\mathbb{F}$-algebra, then $\{a^p, \cdot\} = 0$ so $A^p \subset$ Poisson center & we can view A as a Poisson

$A^{(1)}$-algebra (via $A^{(1)} \twoheadrightarrow A^p$)

• For smooth affine X_0, Poisson center of $\mathbb{F}[T^*X_0]$ coincides w. $\mathbb{F}[T^*X_0]^p \xleftarrow{\sim} \mathbb{F}[T^*X_0]^{(1)}$.

2) Take algebraic group $G =: X_0 \rightsquigarrow G \curvearrowright \mathrm{Vect}(X_0)$ & $\mathfrak{g} = \mathrm{Vect}(X_0)^G$, $G \curvearrowright \mathcal{D}(X_0)$ & $\mathcal{D}(X_0)^G = \mathcal{U}(\mathfrak{g})$.

$(\mathbb{F}[T^*X_0]^{(1)})^G = S(\mathfrak{g})^{(1)}$. So have an algebra embedding

$$S(\mathfrak{g})^{(1)} \hookrightarrow \text{center of } \mathcal{U}(\mathfrak{g}),\ \xi \mapsto \underbrace{\xi^p}_{\in \mathcal{U}(\mathfrak{g})} - \underbrace{\xi^{[p]}}_{\in \mathfrak{g}.}$$

The image is called the p-center.