Lecture 3.

1) Frobenius-constant quantization.
2) Derived equivalences.

1.0) Reminder: $\mathbb{F}$ alg. closed field of char p, X fin. type scheme/$\mathbb{F}$ $\rightsquigarrow X^{(1)}$ scheme over $\mathbb{F}$. If $X \subset \mathbb{A}^n$ closed (given by some equations), then $X^{(1)}$ is given by equations, where we twist coefficients.

Connections between X & $X^{(1)}$:

- They are the same as schemes over $\mathbb{F}_p$ but not over $\mathbb{F}$.
- Have morphism $Fr: X \to X^{(1)}$, its pullback is $f \mapsto f^p$

This morphism is finite, bijective and if X is smooth, Fr is flat of deg $p^{\dim X}$ (exercise).

- If X is defined $/\mathbb{F}_p$, then $X \simeq X^{(1)}$.

Last time we've seen: for a smooth variety X_0, $\mathcal{D}_{X_0}$ can be viewed as Azumaya algebra on $(T^*X_0)^{(1)}$: $Fr: T^*X_0 \to (T^*X_0)^{(1)} \rightsquigarrow$ sheaf of algebras $Fr_* \mathcal{O}_{T^*X_0}$, it's $\mathbb{F}^\times$-equiv ($\mathbb{F}^\times$-action comes from dilations on T^*X_0) & sheaf of Poisson $\mathcal{O}_{(T^*X_0)^{(1)}}$-algebras.

Then $\mathcal{D}_{X_0}$ can be viewed as a filtered quant'n of $Fr_* \mathcal{O}_{T^*X_0}$.

1.1 Frobenius constant quantizations

X smooth variety $/\mathbb{F} \rightsquigarrow Fr: X \to X^{(1)} \rightsquigarrow Fr_* \mathcal{O}_X$ sheaf of Poisson $\mathcal{O}_{X^{(1)}}$-algebras.

For time being let X be affine, $X = Spec(A)$.

Def'n: Let $\mathcal{A}_\hbar$ be formal quant'n of A. We say that $\mathcal{A}_\hbar$ is Frobenius constant if $\exists\ \iota$ w. central image that makes the diagram commutative.

$$\begin{tikzcd} & \mathcal{A}_\hbar \arrow[dr, "/\hbar"] & \\ A^{(1)} \arrow[ur, dashed, "\iota"] \arrow[rr, hook, "a \mapsto a^p"'] & & A \end{tikzcd}$$

Example: $X = T^*\mathbb{A}^n$, $A = \mathbb{F}[x_1, \dots x_n, y_1 \dots y_n]$, $\mathcal{A} = \mathcal{D}(\mathbb{A}^n) \rightsquigarrow$
$\mathcal{A}_\hbar :=$ $\hbar$-adically completed Rees algebra $R_\hbar(\mathcal{A})$

$$\mathcal{A}_\hbar = \mathbb{F}\langle x_i, y_i \rangle[[\hbar]] / \big([x_i, x_j] = [y_i, y_j] = 0,\ [y_i, x_j] = \hbar \delta_{ij}\big)$$

is Frobenius constant w $\iota(x_i) = x_i^p$, $\iota(y_i) = y_i^p$.

Note: $\mathcal{A}_\hbar$ is actually $A^{(1)}[[\hbar]]$-algebra;

Exercise:
- $A^{(1)}[[\hbar]] =$ center of $\mathcal{A}_\hbar$
- $\mathcal{A}_\hbar$ is projective $A^{(1)}[[\hbar]]$-module of $rk = p^{\dim X}$.

Now we no longer assume X is affine $\rightsquigarrow X^{(1)} \times \operatorname{Spec}(\mathbb{F}[\hbar]) \rightsquigarrow$ formal neigh'd of $X^{(1)} \times \{0\}$, to be denoted by $X^{(1)}[[\hbar]]$

Def'n: A Frobenius constant quantization of $\mathcal{O}_X$ is a coherent sheaf of algebras $\mathcal{D}_\hbar$ on $X^{(1)}[[\hbar]]$ satisfying
- $\hbar$ is not a zero divisor in $\mathcal{D}_\hbar$
- $\mathcal{D}_\hbar / (\hbar)$ is comm've.
- have Poisson algebra isom'm $\mathcal{D}_\hbar/(\hbar) \xrightarrow{\sim} Fr_* \mathcal{O}_X$.

Example: • Take $X = T^*X_0$, $\mathcal{D}_{X_0}$ Azumaya algebra on $X^{(1)}$.
$\rightsquigarrow R_\hbar(\mathcal{D}_{X_0})$ sheaf of algebras on $X^{(1)} \times \operatorname{Spec} \mathbb{F}[\hbar] \rightsquigarrow$
$\hbar$-adic completion $\mathcal{D}_\hbar$. It's Frob. const. quant'n.

• For a line bundle $\mathcal{L}$ on $X_0 \rightsquigarrow \mathcal{D}_{X_0,\mathcal{L}} \rightsquigarrow$ Frobenius constant quant'n (the center of $\mathcal{D}_{X_0,\mathcal{L}}$ is identified w. $\mathcal{O}_{(T^*X_0)^{(1)}}$). We'll prove this later.

1.2) Gradings. $\mathbb{F}^\times \curvearrowright X$ s.t. $\{\cdot,\cdot\}$ has $\deg = -1 \rightsquigarrow \mathbb{F}^\times \curvearrowright X^{(1)}$ & $Fr_* \mathcal{O}_X$ is $\mathbb{F}^\times$-equivariant. We can talk about a grading on a Frobenius constant quant'n $\mathcal{D}_\hbar$:

$\mathbb{F}^\times \curvearrowright \mathcal{D}_\hbar$ by alg. autom's making it an $\mathbb{F}^\times$-equiv't sheaf on $X^{(1)}[[\hbar]]$ w. $\deg \hbar = 1$. Also require $\mathcal{D}_\hbar/(\hbar) \xrightarrow{\sim} Fr_* \mathcal{O}_X$ is equiv't.

E.g. $\mathcal{D}_\hbar$ from previous example has a grading.

Goal: exchange $\mathcal{D}_\hbar$ for a filt'd quant'n that is a coherent sheaf of algebras on $X^{(1)}$.

Assumption: Assume X is projective over an affine scheme Y $\rightsquigarrow \mathbb{F}[X]$ is fin. gen'd & graded. Further assume that $\mathbb{F}[X]_i = \{0\}$ $\forall i < 0$.

Fact (algebraization): the $\hbar$-adic completion functor
$$\operatorname{Coh}^{\mathbb{F}^\times}(X^{(1)} \times \operatorname{Spec} \mathbb{F}[\hbar]) \longrightarrow \operatorname{Coh}^{\mathbb{F}^\times}(X^{(1)}[[\hbar]])$$

is equivalence.

$\mathcal{D}_\hbar \in Coh^{\mathbb{F}^\times}(X^{(1)}[[\hbar]]) \rightsquigarrow \mathcal{D}_\hbar^{fin} \in Coh^{\mathbb{F}^\times}(X^{(1)} \times Spec\, \mathbb{F}[\hbar])$, sheaf of algebras $\rightsquigarrow \mathcal{D} := \mathcal{D}_\hbar^{fin}|_{X^{(1)}\times\{1\}}$. This $\mathcal{D}$ can be viewed as filtered Frobenius constant quant'n of $\mathcal{O}_X$.

Prop (Bezrukavnikov-Kaledin) $\mathcal{D}$ is an Azumaya algebra on $X^{(1)}$. (non-split)

Sketch the proof: Assumption $\Rightarrow$ action $\mathbb{F}^\times \curvearrowright X^{(1)}$ is contracting. Enough to check $\forall\ x \in (X^{(1)})^{\mathbb{F}^\times} \Rightarrow \mathcal{D}_x$ is a matrix algebra of rk $p^{\dim X/2}$. $\mathbb{F}[Fr^{-1}(x)]$ is graded (& Poisson) & $\mathcal{D}_x$ is a filtered quant'n of this algebra.

Exercise: • $\mathbb{F}[Fr^{-1}(x)]$ has no nontrivial Poisson ideals.

• $\mathcal{D}_x$ has no nontrivial two-sided ideals.

$\Rightarrow \mathcal{D}_x \simeq Mat_{p^{\dim X/2}}(\mathbb{F})$. $\square$

2) Derived equivalences.

2.1) General result.

$\mathbb{F}$ is an arbitrary field, Y an affine var'y $/\mathbb{F}$, X is projective scheme over Y. Assume X is smooth. Let $\mathcal{R}$ be Azumaya algebra over $X \rightsquigarrow Coh(\mathcal{R}) = \{$sheaves of $\mathcal{R}$-modules that are coherent over $\mathcal{O}_X\}$.

Theorem (Bezrukavnikov-Kaledin): Assume that:

(a) $H^i(X,\mathcal{R})=0 \ \forall i>0$

(b) $\mathcal{A}:=H^0(X,\mathcal{R})$ has finite homological dimension (i.e $\exists n>0$ s.t. $\forall$ $\mathcal{A}$-module has projective resolution of length $\leq n$)

(c) The canonical bundle K_X of X is trivial.

(d) X is connected.

Then the derived global section functor

$$R\Gamma: D^b(Coh\,\mathcal{R}) \longrightarrow D^b(\mathcal{A}\text{-mod}) \text{ is an equivalence.}$$

Proof:

Step 1: $\Gamma: Coh\,\mathcal{R} \longrightarrow \mathcal{A}\text{-mod}$ has left adj't $Loc := \mathcal{R}\otimes_{\mathcal{A}} \bullet$ b/c $\Gamma = Hom_{\mathcal{R}}(\mathcal{R},\bullet)$

Have $R\Gamma: D^b(Coh\,\mathcal{R}) \longrightarrow D^b(\mathcal{A}\text{-mod})$. In general, $LLoc: D^-(\mathcal{A}\text{-mod}) \longrightarrow D^-(Coh\,\mathcal{R})$. Thx to (b) it restricts to D^b and is left adjoint to $R\Gamma$.

Step 2: Claim $R\Gamma\circ LLoc \simeq id_{D^b(\mathcal{A}\text{-mod})}$

$$R\Gamma\circ LLoc(M) = RHom_{\mathcal{R}}(\mathcal{R}, \mathcal{R}\otimes^L_{\mathcal{A}} M) \simeq \overbrace{RHom_{\mathcal{R}}(\mathcal{R},\mathcal{R})}^{R\Gamma(\mathcal{R})}\otimes^L_{\mathcal{A}} M$$

$[R\Gamma(\mathcal{R})=\mathcal{A}$, thx (a)$] = \mathcal{A}\otimes^L_{\mathcal{A}} M = M.$

Step 3: Consider counit $LLoc\circ R\Gamma \longrightarrow id_{D^b(Coh\,\mathcal{R})}$. Want to show it's an isomorphism, equiv. $\forall\, \mathcal{M}\in D^b(Coh\,\mathcal{R})$, the cone, N, of $LLoc\circ R\Gamma(\mathcal{M}) \longrightarrow \mathcal{M}$ is zero. Note $R\Gamma(N)=0 \Leftrightarrow$ $Hom_{D^b(Coh\,\mathcal{R})}(LLoc(?), N)=0.$

Step 4: Notation: $\mathcal{C}:=D^b(Coh\,\mathcal{R})$,

$\mathcal{D} := D^b(A\text{-mod}) \xhookrightarrow[LLoc]{} \mathcal{C}$ – full triangulated subcategory.

$\mathcal{D}^\perp := \{N \in \mathcal{C} \mid Hom_{\mathcal{C}}(?, N) = 0 \ \forall\, ? \in \mathcal{D}\}$.
We want to show that $\mathcal{D}^\perp = \{0\}$.
${}^\perp\mathcal{D} = \{N' \in \mathcal{C} \mid Hom_{\mathcal{C}}(N', ?) = 0 \ \forall\, ? \in \mathcal{D}\}$.
Assume for a moment that $\mathcal{D}^\perp = {}^\perp\mathcal{D}$. Then $\mathcal{C} = \mathcal{D} \oplus \mathcal{D}^\perp$.
Condition $(\alpha) \Rightarrow \mathcal{C}$ is indecomposable. So since $\mathcal{D} \neq \{0\} \Rightarrow$ $\mathcal{D}^\perp$ is zero. This finishes the proof modulo ${}^\perp\mathcal{D} = \mathcal{D}^\perp$.

Step 5: We'll show ${}^\perp\mathcal{D} = \mathcal{D}^\perp$.
Serre duality for smooth projective varieties: if X_0 is smooth projective variety, then

$$RHom_{D^b(Coh X_0)}(\mathcal{F}, \mathcal{G})^* \xrightarrow{\sim} RHom_{D^b(Coh X_0)}(\mathcal{G}, \mathcal{F} \otimes K_{X_0}[\dim X_0])$$

In we work w. $X \to Y$ instead of X_0, just replace $\bullet^*$ w. $\boxed{\underline{RHom_{\mathbb{F}[Y]}(\bullet, \omega_Y)}}$, ω_Y is the dualizing complex in $D^b(\mathbb{F}[Y]\text{-mod})$
Then this generalizes to $Coh(\mathcal{R})$:
equivalence.

$$RHom_{D^b(\mathbb{F}[Y]\text{-mod})}(RHom_{D^b(Coh \mathcal{R})}(\mathcal{F}, \mathcal{G}), \omega_Y) \xrightarrow{\sim}$$

$$RHom_{D^b(Coh \mathcal{R})}(\mathcal{G}, \mathcal{F} \otimes \underbrace{K_X}_{\mathcal{O}_X}[\dim X])$$

So $\left[RHom_{D^b(Coh \mathcal{R})}(\mathcal{F}, \mathcal{G}) = 0 \Leftrightarrow RHom_{D^b(Coh \mathcal{R})}(\mathcal{G}, \mathcal{F}[\dim X]) = 0\right]$
$\Rightarrow {}^\perp\mathcal{D} = \mathcal{D}^\perp$. $\square$

2.2) Derived equivalences from quantizations.

$\mathbb{F}$ is a field. Let X be a symplectic smooth $\mathbb{F}$-variety w. form ω & $\{\cdot,\cdot\}$; $\mathbb{F}^\times \curvearrowright X$ s.t. $\deg\{\cdot,\cdot\} = -1 \rightsquigarrow$ $\mathbb{F}[X]$ is graded Poisson algebra.

Def: Say X is a conical symplectic resolution if

(i) $\mathbb{F}[X]$ is finitely generated $\rightsquigarrow Y := \operatorname{Spec} \mathbb{F}[X]$ & $\pi: X \to Y$.

(ii) π is a projective resolution of singularities.

(iii) $\mathbb{F}[X]_0 = \mathbb{F}$, $\mathbb{F}[X]_i = \{0\}$ $\forall i < 0$.

Rem: X satisfies conditions (c) & (d) of Thm. (exercise)

Ex: G semisimple alg'c group (char $\mathbb{F} = 0$ or not too small) $B \subset G$ Borel, $X = T^*(G/B)$, $Y =$ nilpotent cone in $\mathfrak{g}^*$, $\pi: X \to Y$, Springer resolution.

E.g. $G = SL_2$, $X = T^*\mathbb{P}^1$, $Y = \{(x,y,z) \mid x^2+y^2+z^2 = 0\}$, $\pi: X \to Y$ is blow-up at 0.

Setting: Conical sympl'c resolution $X_{\mathbb{Q}} \xrightarrow{\pi_{\mathbb{Q}}} Y_{\mathbb{Q}}$. These data are defined over a finite localization R of $\mathbb{Z} \rightsquigarrow$ $\pi_R: X_R \to Y_R$. For $p \gg 0$ & alg. closed $\mathbb{F}$ of char p, have $\pi_{\mathbb{F}}: X_{\mathbb{F}} = \operatorname{Spec}(\mathbb{F}) \times_{\operatorname{Spec}(R)} X_R \longrightarrow Y_{\mathbb{F}}$. Since $p \gg 0$, this is still a conical sympl'c resolution.

Suppose $\mathcal{D}$ is a filtered Frobenius constant quantization of $\mathcal{O}_X$ (hence Azumaya algebra on $X^{(1)}$)

Thm: Under assumptions above, $H^i(X^{(1)}_{\mathbb{F}}, \mathcal{D}) = 0 \ \forall\ i > 0$.