Lecture 4: Splitting bundles.

Ref: [BK].

0) Recap: $\mathbb{F}$ base field, $\pi: X \to Y$ conical symplc resol'n
$\mathcal{R}$ Azumaya algebra on X.

Thm (Lec 3): $\mathcal{A} = \Gamma(\mathcal{R})$. If
(i) $H^i(X, \mathcal{R}) = \{0\}$ $\forall$ $i>0$
(ii) $\mathcal{A}$ has finite homological dimension
Then $R\Gamma: D^b(Coh\,\mathcal{R}) \xrightarrow{\sim} D^b(\mathcal{A}\text{-mod})$

1) Derived equivalences from quant'ns.

Observation: If $\mathcal{D}$ is Frobenius constant filtered quant'n of X, then it's Azumaya algebra on $X^{(1)}$.

Example: $X = T^*(G/B)$, $\mathbb{F} = \overline{\mathbb{F}}$, char $\mathbb{F} = p$. Then $\mathcal{D}_{G/B}$ is Azumaya algebra on $X^{(1)}$.

1.1) Condition (i): Suppose $\pi: X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is conical symplc resolution over $\mathbb{Q}$; $\pi: X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is defined over a finite loc'n of $\mathbb{Z}$, denoted by R. For $p \gg 0$ & alg. closed field $\mathbb{F}$ of char $p \rightsquigarrow \pi: X_{\mathbb{F}} \to Y_{\mathbb{F}}$, a conical symplectic resolution.

Thm: Let $\mathcal{D}$, filtered Frobenius constant quant'n of X (so Azumaya algebra on $X^{(1)}_{\mathbb{F}}$). Then $H^i(X^{(1)}_{\mathbb{F}}, \mathcal{D}) = 0$ $\forall$ $i>0$.

Proof:

Step 1: Claim $H^i(X_{\mathbb{F}}, \mathcal{O}) = 0 \ \forall i > 0$.

(i) $H^i(X_{\mathbb{Q}}, \mathcal{O}) = 0 \ \forall i > 0$: this is a special case of Grauert-Riemenschneider theorem: if $\pi: X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ a birational projective morphism & $X_{\mathbb{Q}}$ is smooth, then $R^i\pi_* K_{X_{\mathbb{Q}}} = 0 \ \forall i > 0$.

(ii) $H^i(X_R, \mathcal{O}) = 0$ after finite localization $\forall i > 0$: by (i) $H^i(X_R, \mathcal{O})$ is a torsion R-module. Since π is projective, $H^i(X_R, \mathcal{O})$ is finitely gener'd over $R[Y]$. But $R[Y]$ is finitely generated R-algebra. So $H^i(X_R, \mathcal{O})$ is killed by inverting finitely many primes.

(iii) Have exact sequence $0 \to \mathcal{O}_{X_R} \xrightarrow{p} \mathcal{O}_{X_R} \to \mathcal{O}_{X_{\mathbb{F}_p}} \to 0$

Apply long exact sequence in cohomology $\Rightarrow$
$H^i(X_{\mathbb{F}_p}, \mathcal{O}) = \{0\} \ \forall i > 0 \Rightarrow H^i(X_{\mathbb{F}}, \mathcal{O}) = 0 \ \forall i > 0$.

Step 2: $Fr: X_{\mathbb{F}} \to X_{\mathbb{F}}^{(1)}$ is finite $\Rightarrow H^i(X_{\mathbb{F}}^{(1)}, Fr_* \mathcal{O}_{X_{\mathbb{F}}}) = H^i(X_{\mathbb{F}}, \mathcal{O}_{X_{\mathbb{F}}}) = 0 \ \forall i > 0$.

Recall from Lec 3, have an $\mathbb{F}^\times$-equiv't coherent sheaf $\mathcal{D}_\hbar^{fin}$ on $X_{\mathbb{F}}^{(1)} \times \operatorname{Spec} \mathbb{F}[\hbar]$ s.t.

- $\mathcal{D}_\hbar^{fin} / \hbar \mathcal{D}_\hbar^{fin} \xrightarrow{\sim} Fr_* \mathcal{O}_{X_{\mathbb{F}}}$ (1)

- $\mathcal{D}_\hbar^{fin} / (\hbar - 1) \mathcal{D}_\hbar^{fin} \xrightarrow{\sim} \mathcal{D}$.

Enough to show $H^i(X_{\mathbb{F}} \times \operatorname{Spec} \mathbb{F}[\hbar], \mathcal{D}_\hbar^{fin}) = 0 \ \forall i > 0$.

By long exact sequence for (1):

$$\to \underbrace{H^i(\mathcal{D}_\hbar^{fin})} \xrightarrow{\hbar} H^i(\mathcal{D}_\hbar^{fin}) \longrightarrow H^i(Fr_* \mathcal{O}_{X_{\mathbb{F}}}) = \{0\} \qquad (2)$$

finitely generated, graded $\mathbb{F}[Y^{(1)}][\hbar] = \mathbb{F}[X^{(1)}][\hbar]$ -module. But the algebra is positively graded. By graded Nakayama + (2) $\Rightarrow H^i(\mathcal{D}_\hbar^{fin}) = \{0\}$ $\forall i>0$ $\square$

Exercise: Prove that $H^0(X^{(1)}, \mathcal{D})$ is a filtered quantization of $\mathbb{F}[Y]$ ($= \mathbb{F}[X]$).

1.2) Finite homological dimension At least in examples we can find $\mathcal{D}$ s.t. $\mathcal{A} = \Gamma(\mathcal{D})$ have finite homological dimension. Sometimes ($X = T^*(G/B)$ & X = resolution of sympl'c quotient sing'y e.g. $X = \mathrm{Hilb}_n(\mathbb{F}^2)$) can find such $\mathcal{D}$ directly. In other cases can argue by reduction from char 0:

Quantizations of $X_{\mathbb{C}}$ classified by $H^2(X, \mathbb{C})$ ($\neq \{0\}$)
For $\mathcal{D}_{\mathbb{C},\lambda}$ corresp. to a Zariski generic $\lambda \in H^2(X, \mathbb{C})$, $\Gamma(\mathcal{D}_{\mathbb{C},\lambda})$ have finite homological dimension. For $\lambda \in H^2(X, \mathbb{Q})$ can reduce $\mathcal{D}_{\mathbb{C},\lambda}$ & $\Gamma(\mathcal{D}_{\mathbb{C},\lambda})$ mod $p \gg 0$, in all examples the reduction of $\mathcal{D}_{\mathbb{C},\lambda}$ is Frobenius-constant. One can essentially show that reducing $\Gamma(\mathcal{D}_{\mathbb{C},\lambda})$ mod $p \gg 0$ preserves the homological dimension.

Conclusion: Essentially always can find Frobenius const. quant'n $\mathcal{D}$ s.t $R\Gamma: D^b(\mathrm{Coh}\,\mathcal{D}) \xrightarrow{\sim} D^b(\mathcal{A}\text{-mod})$ ($\mathcal{A} = \Gamma(\mathcal{D})$)

But how useful is this. Often we care about $\mathcal{A}$. For $T^*(G/B)$, $\mathcal{A}$ is a (Harish-Chandra) central red'n of $U(\mathfrak{g})$. $Coh\,\mathcal{D}$ - not so much...

On the other hand if Azumaya algebra $\mathcal{R}$ that is <u>split</u>: $\mathcal{R} = End(V)$, where V is vector bundle, then

$$Coh(\mathcal{R}) \xleftarrow[V\otimes_{\mathcal{O}_X}\bullet]{\sim} Coh(X)$$

Issue: a Frobenius constant quantization doesn't split.

Fix: Still $\mathcal{D}$ splits "somewhere", this allows to produce a split Azumaya algebra on $X^{(1)}_{\mathbb{F}}$ (and even $X_{\mathbb{C}}$) that will produce a derived equivalence.

2) Splitting bundles.

Reminder: if Z is any variety, $z\in Z \rightsquigarrow \hat{\mathcal{O}}_{Z,z}$ complete local ring $\rightsquigarrow Z^{\wedge z} := Spec\,\hat{\mathcal{O}}_{Z,z}$. Restriction of any Azumaya algebra to this subscheme splits.

Fact: In cases of interest, the restriction of Frobenius constant quantization $\mathcal{D}$ to

$$Y^{(1),\wedge y}\times_{Y^{(1)}} X^{(1)} =: X^{(1),\wedge y}$$

splits $\forall y\in Y^{(1)}$.

small neigh'd of fiber of y.

This follows from combining Kubrak-Travkin & Bogdanova-Vologodsky (for certain Frobenius-constant quant'ns that in examples are reduced mod p from char 0 and so include $\mathcal{D}$ w. $\Gamma(\mathcal{D})$ has finite homological dimension).

Take $y=0 \rightsquigarrow X^{(1),\wedge_0}$. Let $\mathcal{E}'_{\mathbb{F}}$ be a splitting bundle for $\mathcal{D}|_{X^{(1),\wedge_0}}$ (defined up to twisting w. line bundle). $\mathrm{End}(\mathcal{E}'_{\mathbb{F}})$

- $\mathcal{A}^{\wedge_0} = \Gamma(\mathcal{D})^{\wedge_0} = [\text{formal function thm}] = \Gamma(\mathcal{D}|_{X^{(1),\wedge_0}}) = \mathrm{End}(\mathcal{E}'_{\mathbb{F}})$
- $\mathrm{Ext}^i(\mathcal{E}'_{\mathbb{F}}, \mathcal{E}'_{\mathbb{F}}) = H^i(X,\mathcal{D})^{\wedge_0} = 0.$
- $R\Gamma: D^b(\mathrm{Coh}\,\mathcal{D}) \xrightarrow{\sim} D^b(\mathcal{A}\text{-mod}) \Rightarrow$

$$\begin{tikzcd} R\Gamma: D^b(\mathrm{Coh}\,\mathcal{D}|_{X^{(1),\wedge_0}}) \arrow[r, "\sim"] & D^b(\mathcal{A}^{\wedge_0}\text{-mod}) \\ D^b(\mathrm{Coh}\, X^{(1),\wedge_0}) \arrow[u, "\simeq", "\mathcal{E}'_{\mathbb{F}}\otimes\bullet"'] & \end{tikzcd}$$

2.1) Extension to $X^{(1)}_{\mathbb{F}}$. Want to extend $\mathcal{E}'_{\mathbb{F}}$ to a vector bundle over $X^{(1)}_{\mathbb{F}}$.

$\mathbb{F}^\times \curvearrowright X_{\mathbb{F}} \rightsquigarrow \mathbb{F}^\times \curvearrowright X^{(1)}_{\mathbb{F}} \rightsquigarrow \mathbb{F}^\times \curvearrowright X^{(1),\wedge_0}_{\mathbb{F}}$.

Fact (Vologodsky): Since $\mathrm{Ext}^1(\mathcal{E}'_{\mathbb{F}}, \mathcal{E}'_{\mathbb{F}}) = \{0\}$, $\mathcal{E}'_{\mathbb{F}}$ has an $\mathbb{F}^\times$-equiv't structure.

We'll use this to extend $\mathcal{E}'_{\mathbb{F}}$ to $X^{(1)}_{\mathbb{F}}$. Recall have proj've resol'n morphism $\pi: X^{(1)}_{\mathbb{F}} \to Y^{(1)}_{\mathbb{F}}$, $\mathbb{F}^\times$-equiv't; $\mathbb{F}[Y^{(1)}]$ is positively graded, equivalently, $\mathbb{F}^\times$ contracts $X^{(1)}$ to $\pi^{-1}(0)$.

$\mathbb{F}^\times \curvearrowright \mathcal{E}'_{\mathbb{F}} \rightsquigarrow \mathbb{F}^\times \curvearrowright \mathcal{A}^{\wedge_0} = End(\mathcal{E}'_{\mathbb{F}})$.

$\rightsquigarrow \tilde{A}_{\mathbb{F}} = \mathbb{F}^\times$-finite part of $\mathcal{A}^{\wedge_0}$, graded algebra over $\mathbb{F}[\mathcal{Y}^{(1)}]$.

Exercise: • The grading on $\tilde{A}_{\mathbb{F}}$ is bounded from below.

• $\tilde{A}_{\mathbb{F}}^{\wedge_0} \longrightarrow \mathcal{A}^{\wedge_0}$.

• Completion functor $\bullet^{\wedge_0}$: $\tilde{A}_{\mathbb{F}}\text{-mod}^{\mathbb{F}^\times} \xrightarrow{\sim} \mathcal{A}^{\wedge_0}\text{-mod}^{\mathbb{F}^\times}$

Fact: Restriction to $X_{\mathbb{F}}^{\wedge_0}$ defines an equivalence:

$$Coh^{\mathbb{F}^\times}(X_{\mathbb{F}}^{(1)}) \longrightarrow \underset{\mathcal{E}'_{\mathbb{F}}}{Coh^{\mathbb{F}^\times}}{}^{\Psi}(X_{\mathbb{F}}^{(1),\wedge_0}).$$

Let $\mathcal{E}_{\mathbb{F}} \in Coh^{\mathbb{F}^\times}(X_{\mathbb{F}}^{(1)})$ be the image of $\mathcal{E}'_{\mathbb{F}}$ under the equivalence.

Exercise: $End(\mathcal{E}_{\mathbb{F}}) \xrightarrow{\sim} \tilde{A}_{\mathbb{F}}$ & $Ext^i(\mathcal{E}_{\mathbb{F}}, \mathcal{E}_{\mathbb{F}}) = 0 \ \forall i > 0$.

Lemma: $R\Gamma(\mathcal{E}_{\mathbb{F}} \otimes \cdot): D^b(Coh\, X_{\mathbb{F}}^{(1)}) \xrightarrow{\sim} D^b(\tilde{A}_{\mathbb{F}}\text{-mod})$

Proof:

$\mathbb{F}^\times$-equiv't

$$R\Gamma(\mathcal{E}'_{\mathbb{F}} \otimes \cdot): D^b(Coh\, X^{(1),\wedge_0}) \xrightarrow{\sim} D^b(\mathcal{A}^{\wedge_0}\text{-mod})$$

$$\Updownarrow$$

$$R\Gamma(\mathcal{E}'_{\mathbb{F}} \otimes \cdot): D^b(Coh^{\mathbb{F}^\times} X^{(1),\wedge_0}) \xrightarrow{\sim} D^b(\mathcal{A}^{\wedge_0}\text{-mod}^{\mathbb{F}^\times})$$

$$\updownarrow\wr \qquad\qquad \updownarrow\wr$$

$$R\Gamma(\mathcal{E}_{\mathbb{F}} \otimes \cdot): D^b(Coh^{\mathbb{F}^\times} X^{(1)}) \xrightarrow{\sim} D^b(\tilde{A}_{\mathbb{F}}\text{-mod}^{\mathbb{F}^\times})$$

$$\Rightarrow$$

$$R\Gamma(\mathcal{E}_{\mathbb{F}} \otimes \cdot): D^b(Coh\, X^{(1)}) \xrightarrow{\sim} D^b(\tilde{A}_{\mathbb{F}}\text{-mod})$$

$\square$

2.2) Lift to characteristic 0.

$X^{(1)}_{\mathbb{F}} \xrightarrow{\sim} X_{\mathbb{F}}$ so we can view $\mathcal{E}_{\mathbb{F}}$ as a vector bundle over $X_{\mathbb{F}}$.
It's defined over $\mathbb{F}_q \rightsquigarrow \mathcal{E}_{\mathbb{F}_q}$ over $X_{\mathbb{F}_q}$. Let R be an alg'c extension of $\mathbb{Z}$ s.t

- $R \twoheadrightarrow \mathbb{F}_q$
- X is defined over R and is nice.

$\rightsquigarrow X_{\mathbb{F}_q}$ is a closed subscheme of X_R

Let $R^{\wedge q} :=$ completion of R at $\ker[R \twoheadrightarrow \mathbb{F}_q]$
Consider $X_R^{\wedge q} :=$ formal neigh'd of $X_{\mathbb{F}_q}$ in X_R, a formal scheme.

Since $\mathrm{Ext}^i(\mathcal{E}_{\mathbb{F}_q}, \mathcal{E}_{\mathbb{F}_q}) = 0$ for $i = 1, 2$, $\mathcal{E}_{\mathbb{F}_q}$ deforms uniquely to a vector bundle over $X_R^{\wedge q}$ & the deform'n is $\mathbb{G}_m$-equiv't.
Since $\mathbb{G}_m$-action is contracting we can algebrize this deform'n getting an equiv't vector bundle on $X_{R^{\wedge q}}$.

$R^{\wedge q} \hookrightarrow \mathbb{C} \rightsquigarrow$ get vector bundle $\mathcal{E}_{\mathbb{C}}$ on $X_{\mathbb{C}}$.

Properties:
- It's $\mathbb{C}^\times$-equiv't
- $\tilde{A}_{\mathbb{C}} = \mathrm{End}(\mathcal{E}_{\mathbb{C}})$ has finite homological dim'n
- $\mathrm{Ext}^i(\mathcal{E}_{\mathbb{C}}, \mathcal{E}_{\mathbb{C}}) = 0 \ \forall i > 0$
- $R\Gamma(\mathcal{E}_{\mathbb{C}} \otimes \bullet): D^b(\mathrm{Coh}\, X_{\mathbb{C}}) \xrightarrow{\sim} D^b(\tilde{A}_{\mathbb{C}}\text{-mod})$

2.3) Identifying $\tilde{\mathcal{A}}_{\mathbb{C}}$.

Notice: $\mathcal{E}_{\mathbb{C}}$ depends on p, $\mathrm{rk}\, \mathcal{E}_{\mathbb{C}} = p^{\dim X/2}$. However by picking direct summands of $\mathcal{E}_{\mathbb{C}}$ w. diff'nt multiplicities in known examples can achieve that $\mathcal{E}_{\mathbb{C}}$ is independent of p.

Can describe $\tilde{\mathcal{A}}_{\mathbb{C}}$ in the following cases:

(I) $Y = V/\Gamma$, V is sympl. vector space, $\Gamma \subset Sp(V)$ finite gr'p.
Then $\tilde{\mathcal{A}}_{\mathbb{C}} = \mathbb{C}[V] \# \Gamma$

(II) X is smooth Coulomb branch of a gauge th'y (constructed BFN). $\tilde{\mathcal{A}}_{\mathbb{C}}$ was described by Webster. Possible X include hypertoric var's & finite & affine type A Nakajima quiver var's.

Fact (Kaledin): $\tilde{\mathcal{A}}_{\mathbb{C}}$ depends only on Y but not on X.
For two symplic resol'ns X, X' of Y have:

$$D^b(\mathrm{Coh}\, X) \xrightarrow{\sim} D^b(\tilde{\mathcal{A}}_{\mathbb{C}}\text{-mod}) \xleftarrow{\sim} D^b(\mathrm{Coh}\, X')$$

This is a special case of K-equivalence $\Rightarrow$ D-equivalence conjecture.