Lecture 5. updated 10/8
The case of Springer resolution.

Ref: [BMR]

1) Localization theorems.

$\mathbb{F}$ alg. closed field, G semisimple simply conn'd alg. gr'p/$\mathbb{F}$. Assume that char $\mathbb{F}=0$ or $p>h$ (h is the Coxeter number, $h=n$ for SL_n). Then $\mathfrak{g}$ admits a non-degenerate G-inv't symm'c form $(\cdot,\cdot) \rightsquigarrow \mathfrak{g}\simeq\mathfrak{g}^*$.

Take $Y:=$ nilpotent cone $\mathcal{N}\subset\mathfrak{g}^*(\simeq\mathfrak{g})$, $G\curvearrowright\mathcal{N}$.

Fact: • Y is irreducible & normal

• $G\curvearrowright Y$ has finitely many orbits.

$B\subset G$, Borel subgroup, $\rightsquigarrow \mathcal{B}:=G/B=\{$Borel subalg's in $\mathfrak{g}\}$

$X=T^*(G/B)\simeq\{(x,\mathfrak{b}')\in\mathfrak{g}\times\mathcal{B} \mid x\in[\mathfrak{b}',\mathfrak{b}']\}$

Springer morphism $\pi: X=T^*(G/B)\to Y=\mathcal{N}$, $\pi(x,\mathfrak{b}')=x$.

Facts: • X is a resolution of sing's of Y (via π) $\Rightarrow$ $\pi^*: \mathbb{F}[Y]\xrightarrow{\sim}\mathbb{F}[X]$.

• $H^i(X,\mathcal{O})=0$ $\forall i>0$ – have seen when char $\mathbb{F}=0$ or $\gg 0$.

Sheaves of twisted diff'l operators on G/B. Since G is simply connected, $\mathrm{Pic}(\mathcal{B})\xleftrightarrow{\sim}\mathfrak{X}(B)$, character lattice

$$\mathcal{O}(\lambda)\longleftarrow\!\shortmid\ \lambda$$

$\rightsquigarrow \mathcal{D}^\lambda_{\mathcal{B}} := \mathcal{D}_{\mathcal{B}}(\mathcal{O}(\lambda))$.

When char $\mathbb{F}=0$, $\mathcal{D}^\lambda_{\mathcal{B}}$'s are pairwise distinct

when char $\mathbb{F}=p>0$, $\mathcal{D}^\lambda_{\mathcal{B}} \simeq \mathcal{D}^{\lambda'}_{\mathcal{B}} \Leftrightarrow \lambda'-\lambda \in p\mathfrak{X}(B)$.

$\mathcal{D}^\lambda_{\mathcal{B}}$ is obtained by quantum Hamiltonian reduction for the action of T (max'l torus in B) on $\mathcal{D}_{G/U}$ ($U = R_u(B) = (B,B)$)

$G/U \curvearrowleft T \rightsquigarrow \Phi: \mathfrak{t} \longrightarrow \Gamma(\mathcal{D}_{G/U})$, $\xi \mapsto \xi_{G/U}$.

$\varpi: G/U \xrightarrow[/T]{} G/B$

$$\mathcal{D}^\lambda_{\mathcal{B}} = \mathcal{D}_{G/U} /\!\!/\!\!/_\lambda T := \varpi_*\left[\mathcal{D}_{G/U} / \mathcal{D}_{G/U}\{\xi_{G/U} - \langle\lambda,\xi\rangle \mid \xi\in\mathfrak{t}\}\right]^T.$$

$$= (\varpi_*\mathcal{D}_{G/U})^T / (\varpi_*\mathcal{D}_{G/U})^T\{\xi_{G/U} - \langle\lambda,\xi\rangle \mid \xi\in\mathfrak{t}\}.$$

quant'n of X see (†) on page 6.

Quantizations of $Y=\mathcal{N}$: "Harish-Chandra" center $\mathcal{U}(\mathfrak{g})^G \subset \mathcal{U}(\mathfrak{g})$ (= full center in char = 0): $\mathcal{U}(\mathfrak{g})^G \xrightarrow{\sim} \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$

$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$, $w\cdot\lambda = w(\lambda+\rho)-\rho$.

$\lambda \rightsquigarrow$ max'l ideal $\mathfrak{m}_\lambda \subset \mathcal{U}(\mathfrak{g})^G = \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$ - all el'ts vanishing at λ

Def: $\mathcal{U}_\lambda = \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{m}_\lambda$ - algebra.

Fact: $\mathcal{U}_\lambda$ is a filtered quantization of $\mathbb{F}[\mathcal{N}]$.

Note $G \curvearrowright \mathcal{D}^\lambda_{\mathcal{B}}$ is Hamiltonian w. quantum comoment map Φ

$\lambda=0$: $\Phi: \mathfrak{g} \longrightarrow \Gamma(\mathcal{D}_{\mathcal{B}})$, $\xi \mapsto \xi_{\mathcal{B}}$

any λ: Φ descends from $\xi \mapsto \xi_{G/U}: \mathfrak{g} \to \Gamma(\mathcal{D}_{G/U}) \rightsquigarrow$ algebra homomorphisms $\Phi_\lambda: \mathcal{U}(\mathfrak{g}) \longrightarrow \Gamma(\mathcal{D}^\lambda_{G/B})$, $\Phi_{\mathfrak{t}}: \mathcal{U}(\mathfrak{g}) \longrightarrow \Gamma(\mathcal{D}_{G/U})^T$ so that Φ_λ is the composition of $\Phi_{\mathfrak{t}}$ w. $\Gamma(\mathcal{D}_{G/U})^T \longrightarrow \Gamma(\mathcal{D}^\lambda_{G/B})$

Prop'n : Φ descends to $\mathcal{U}_\lambda \xrightarrow{\sim} \Gamma(\mathcal{D}^\lambda_{G/B})$ | specialization to $\lambda \in \mathfrak{t}^*$

Rem: $H^i(\mathcal{B}, \mathcal{D}^\lambda_{\mathcal{B}}) = 0 \ \forall i > 0$.

Proof of Prop'n:

Step 1: $\Phi: \mathcal{U}(\mathfrak{g}) \twoheadrightarrow \Gamma(\mathcal{D}^\lambda_{G/B})$ – homomorphism of filtered algebras, $\operatorname{gr}\Phi: S(\mathfrak{g}) \longrightarrow \operatorname{gr}\Gamma(\mathcal{D}^\lambda_{G/B}) = [H^1(\mathcal{B}, \mathcal{D}^\lambda_{\mathcal{B}}) = 0] =$
$= \Gamma(\operatorname{gr}\mathcal{D}^\lambda_{G/B}) = \mathbb{F}[X] = \mathbb{F}[\mathcal{N}]$.

Can see $\operatorname{gr}\Phi$ is the projection, so it's surjective $\Rightarrow \Phi$ is surjective.

Step 2: Φ is G-invariant & $\mathcal{N}$ has open G-orbit $\Rightarrow \mathbb{F}[\mathcal{N}]^G = \mathbb{F}$
$\operatorname{gr}\Gamma(\mathcal{D}^\lambda_{G/B})^G \hookrightarrow \mathbb{F}[\mathcal{N}]^G \Rightarrow \Gamma(\mathcal{D}^\lambda_{G/B})^G = \mathbb{F}$ so $\Phi_\lambda: \mathcal{U}(\mathfrak{g})^G \longrightarrow \mathbb{F}$.

Assume char $\mathbb{F} = 0$. Claim: $\Phi_\lambda|_{\mathcal{U}(\mathfrak{g})^G}$ factors through λ. One can show that $[\Gamma(\mathcal{D}_{G/U})^T]^G \xleftarrow{\sim} S(\mathfrak{t})$ so $\Phi_\lambda: \mathcal{U}(\mathfrak{g})^G \to \mathbb{F}$ factors as $\Phi_{\mathfrak{t}}: \mathcal{U}(\mathfrak{g})^G \longrightarrow S(\mathfrak{t}) \longrightarrow \mathbb{F}$, where the 2nd arrow is specialization to λ. A consequence is that $\Phi_\lambda|_{\mathcal{U}(\mathfrak{g})^G}$ depends polynomially on λ So it's enough to prove our claim for dominant λ b/c such elements are Zariski dense in $\mathfrak{t}^*$. Then $\mathcal{D}^\lambda_{G/B}$ acts on $\mathcal{O}(\lambda)$
If λ is dominant, Borel-Weil thm $\Rightarrow \Gamma(\mathcal{O}(\lambda)) = L(\lambda)$ – irrep. w. highest wt λ. The action of $\mathcal{U}(\mathfrak{g})^G$ on $L(\lambda)$ is via λ.

So
$$\begin{tikzcd} \Phi: \mathcal{U}(\mathfrak{g}) \arrow[rr] \arrow[dr, two heads] & & \Gamma(\mathcal{D}^\lambda_{\mathcal{B}}) \\ & \mathcal{U}_\lambda \arrow[ur] & \end{tikzcd}$$

Associated graded of $\mathcal{U}_\lambda \longrightarrow \Gamma(\mathcal{D}^\lambda_{\mathcal{B}})$ is $\mathbb{F}[\mathcal{N}] \xrightarrow{id} \mathbb{F}[\mathcal{N}]$

So $\mathcal{U}_\lambda \xrightarrow{\sim} \Gamma(\mathcal{D}^\lambda_{G/B})$.

Step 3: For char $\mathbb{F} = p \gg 0$ the result will follow. W/o this assumption we can use $\Gamma(\mathcal{O}(\lambda))$ = dual Weyl module w. highest wt λ. The proof can be deduced from there. $\square$

So can consider $\Gamma: Coh(\mathcal{D}^\lambda_{G/B}) \longrightarrow \mathcal{U}_\lambda\text{-mod}$ &
$R\Gamma: D^b(Coh\, \mathcal{D}^\lambda_{G/B}) \longrightarrow D^b(\mathcal{U}_\lambda\text{-mod})$.

Thm (Beilinson-Bernstein). Assume char $\mathbb{F}=0$:

- Γ is equivalence $\Leftrightarrow$ λ is dominant.
- If $\lambda+\rho$ is dominant $\Leftrightarrow$ Γ is exact.
- $\lambda+\rho$ is regular ($\langle \lambda+\rho, \alpha^\vee\rangle \neq 0$ $\forall$ coroot $\alpha^\vee$) $\Leftrightarrow$

$R\Gamma$ is an equivalence.

Thm (BMR). Still assume char $p>h$:
If $\lambda+\rho$ is regular mod p ($\langle \lambda+\rho, \alpha^\vee\rangle \neq 0$ in $\mathbb{F}_p$ $\forall \alpha^\vee$), then $R\Gamma$ is an equivalence.
Have seen this for $p \gg 0$ (Lectures 3 & 4), there's general proof.

2) Azumaya algebras from twisted differential operators

X_0 smooth variety over $\mathbb{F}$, $\mathbb{F}=\overline{\mathbb{F}}$, char $\mathbb{F} = p > 0$.
We've seen $\mathcal{D}_{X_0}$ is Azumaya algebra over $X^{(1)}$, $X = T^*X_0$; pick line bundle $\mathcal{L}$ on X_0 $\rightsquigarrow$ sheaf of twisted diff. operators, $\mathcal{D}_{X_0}(\mathcal{L})$.

Thm: $\mathcal{D}_{X_0}(\mathcal{L})$ is Azumaya algebra on $X^{(1)}$, Morita equivalent to $\mathcal{D}_{X_0}$ ($\Longleftrightarrow$ $\mathcal{D}_{X_0}(\mathcal{L}) \otimes \mathcal{D}_{X_0}^{opp}$ splits).

Assume H is connected algebraic group acting on X_0, $X = T^*X_0$.
Classical comoment map $\varphi: S(\mathfrak{h}) \to \mathbb{F}[X] \rightsquigarrow \varphi^{(1)}: S(\mathfrak{h}^{(1)}) \to \mathbb{F}[X^{(1)}]$.
Quantum comoment map $\Phi: \mathcal{U}(\mathfrak{h}) \to \mathcal{D}(X_0)$ - global diff. op-rs.
$\mathbb{F}[X^{(1)}] \hookrightarrow \mathcal{D}(X_0)$, for $\xi \in Vect(X_0^{(1)})$, $\Phi(\xi) = \xi^p - \xi^{[p]}$.
$S(\mathfrak{h}^{(1)}) \hookrightarrow \mathcal{U}(\mathfrak{h})$, $\eta \mapsto \eta^p - \eta^{[p]}$.

Lemma: The following diagram is commutative:

$$\begin{tikzcd}
S(\mathfrak{h}^{(1)}) \arrow[r, "\varphi^{(1)}"] \arrow[d] & \mathbb{F}[X^{(1)}] \arrow[d] \\
\mathcal{U}(\mathfrak{h}) \arrow[r, "\Phi"] & \mathcal{D}(X_0)
\end{tikzcd}$$

Proof: One needs to show: $\xi \mapsto \xi_{X_0}: \mathfrak{h} \to Vect(X_0)$ satisfies $(\xi^{[p]})_{X_0} = (\xi_{X_0})^{[p]}$ — exercise

Hint: consider $a: H \times X_0 \to X_0$ gives well-defined map on <u>some</u> vector fields, namely right-invariant vector fields on H can be viewed as vector fields on $H \times X_0$ and da is well-defined on them $\rightsquigarrow \mathfrak{h} \to Vect(X_0)$, it's $\xi \mapsto \xi_{X_0}$.

$\square$

Proof of Thm: Take a torus T and let $\tilde{X}_0 \xrightarrow{\varpi} X_0$ be principal T-bundle (e.g. $T = \mathbb{F}^\times$; $\tilde{X}_0 \to X_0$ comes from $\mathcal{L}$).

Pick $\lambda \in \mathfrak{X}(T) \rightsquigarrow \mathcal{D}_{\tilde{X}_0} /\!/\!/_\lambda T =$

$= \varpi_* [\mathcal{D}_{\tilde{X}_0} / \mathcal{D}_{\tilde{X}_0} \{\xi_{\tilde{X}_0} - \langle \lambda, \xi \rangle\}]^T = [T$ is linearly reductive, & $\xi_{\tilde{X}_0} - \langle\lambda,\xi\rangle$ are T-invariant$] = (\varpi_* \mathcal{D}_{\tilde{X}_0})^T / (\varpi_* \mathcal{D}_{\tilde{X}_0})^T \{\xi_{\tilde{X}_0} - \langle \lambda, \xi\rangle\}$. (†)

$\mathcal{D}_{\tilde{X}_0}$ is a T-equiv't sheaf of algebras on $\tilde{X}^{(1)} \curvearrowleft T$. So $(\varpi_* \mathcal{D}_{\tilde{X}_0})^T$ can be viewed as a sheaf of $\tilde{X}^{(1)}/T^{(1)}$.

(for $X_0 = G/B$, $\tilde{X}_0 = G/U$, $\tilde{X} = G \times^U \mathfrak{b} \Rightarrow \tilde{X}/T = G \times^B \mathfrak{b}$, it's a scheme over $\mathfrak{t}$, the fiber over 0 in $G \times^B \mathfrak{n} = X$, the other fibers are twisted cotangent bundles) It's similar for $\tilde{X}/T$ in general.

The map $\xi \mapsto \xi_{\tilde{X}_0}$ makes $(\varpi_* \mathcal{D}_{\tilde{X}_0})^T$ into a sheaf of $S(\mathfrak{t})$-algebras on $\tilde{X}^{(1)}/T^{(1)}$. Then $S(\mathfrak{t}^{(1)}) \longrightarrow \mathcal{O}_{\tilde{X}^{(1)}/T^{(1)}}$, $S(\mathfrak{t}) = \mathcal{U}(\mathfrak{t})$. By lemma $(\varpi_* \mathcal{D}_{\tilde{X}_0})^T$ is a sheaf of algebras on

$$\tilde{X}^{(1)}/T^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$$

The map $\mathfrak{t}^* \longrightarrow \mathfrak{t}^{*(1)}$ comes from $S(\mathfrak{t}^{(1)}) \to \mathcal{U}(\mathfrak{t})$, $\xi \mapsto \xi^p - \xi^{[p]}$. For $\xi \in \mathfrak{t}_{\mathbb{F}_p}$, have $\xi^{[p]} = \xi$ (if $T = \mathbb{F}^\times$, the vector field corresp. to $1 \in \mathfrak{t}$ is $z\partial_z$). So if we choose a basis of $\mathfrak{t}$ from a basis in $\mathfrak{X}(T)$, then $\mathfrak{t}^* \to \mathfrak{t}^{*(1)}$ is

AS: $(z_1, \dots z_n) \mapsto (z_1^p - z_1, \dots z_n^p - z_n)$.

In particular, if $z_i \in \mathbb{F}_p \ \forall i$, then $AS(z_1, \dots z_n) = 0$. In particular, the image of $\lambda \in \mathfrak{t}^*$ is $0 \in \mathfrak{t}^{*(1)}$.

$\mathcal{D}_{\tilde{X}_0} /\!/\!/_\lambda T$ is the specialization of $(\varpi_* \mathcal{D}_{\tilde{X}_0})^T$ to $\lambda \in \mathfrak{t}^*$. This is a sheaf of algebras over

$$\tilde{X}^{(1)}/T^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^* \times_{\mathfrak{t}^*} \{\lambda\} = \tilde{X}^{(1)}/T^{(1)} \times_{\mathfrak{t}^{*(1)}} \{AS(\lambda)\}$$

$$= [AS(\lambda) = 0] = \tilde{X}^{(1)}/T^{(1)} \times_{\mathfrak{t}^{*(1)}} \{0\} = X^{(1)}.$$

To show that $\mathcal{D}_{\tilde{X}_0} /\!/\!/_\lambda T$ is Azumaya, note this is a local property. Pick $X_0' \subset X_0$ s.t. $\tilde{X}_0' := X_0' \times_{X_0} \tilde{X}_0 \xrightarrow{\sim} T \times X_0'$. Then

$$\mathcal{D}_{\tilde{X}_0} /\!/\!/_\lambda T|_{X_0'} = \mathcal{D}_{\tilde{X}_0'} /\!/\!/_\lambda T \simeq \mathcal{D}_{X_0'}, \text{ which is Azumaya.}$$

Recall $\lambda \in \mathfrak{X}(T) \rightsquigarrow \varpi_* [\mathcal{D}_{\tilde{X}_0} / \mathcal{D}_{\tilde{X}_0} \{\xi_{\tilde{X}_0}\}]^{T, \lambda}$ - λ-semi-invariants. In earlier lecture, this is a bimodule over $\mathcal{D}_{\tilde{X}_0} /\!/\!/_\lambda T$ - $\mathcal{D}_{\tilde{X}_0} /\!/\!/_0 T$. Over X_0', this is a regular bimodule. So it gives a Morita equivalence between $\mathcal{D}_{\tilde{X}_0} /\!/\!/_\lambda T$ and $\mathcal{D}_{X_0}$. (if $\mathcal{L}$ is the line bundle obtained from λ by descent, then $\mathcal{D}_{\tilde{X}_0} /\!/\!/_\lambda T = \mathcal{L} \otimes \mathcal{D}_{X_0} \otimes \mathcal{L}^{-1}$ & the bimodule is $\mathcal{L} \otimes \mathcal{D}_{X_0}$) $\square$