Lecture 6.

Ref: [BMR].

Lec 4: discussed splitting for Azumaya algebras on $X^{(1)}$ arising from Frobenius constant quantizations of X, where X is a conical symplic resolution

Lec 5: Considered quantization $\mathcal{D}^{\lambda}_{G/B}$ of $X = T^*(G/B)$

Today: discuss splitting for $\mathcal{D}^{\lambda}_{G/B}$, its applications & properties of the splitting bundle.

1) Splitting: $\mathbb{F}$ alg. closed field of char $p>0$, X_0 smooth var'y, $X = T^*X_0 \rightsquigarrow$ Frobenius constant quant'n $\mathcal{D}_{X_0}(\mathcal{L})$, $\mathcal{L}$ is a line bundle on X_0: $\mathcal{D}_{X_0}(\mathcal{L})$ is an Azumaya algebra on $X^{(1)}$.

Easy observation: $\mathcal{D}_{X_0}(\mathcal{L})|_{X_0^{(1)}}$ splits & for splitting bundle can take $Fr_*\mathcal{L}$:

- $\mathcal{D}_{X_0}(\mathcal{L}) \curvearrowright \mathcal{L} \rightsquigarrow \mathcal{D}_{X_0}(\mathcal{L})|_{X_0^{(1)}} \curvearrowright Fr_*\mathcal{L}$
- $\forall x \in X_0^{(1)}$, this action gives $\mathcal{D}_{X_0}(\mathcal{L})|_x \xrightarrow{\sim} End((Fr_*\mathcal{L})_x)$.

Specialize: $X_0 = G/B$, $Y = \mathcal{N}$ - nilpotent cone, $\pi: X \to Y$ is Springer resolution

For $\lambda \in \mathfrak{h}^* \rightsquigarrow$ central reduction $\mathcal{U}_\lambda$ - algebra over $\mathbb{F}[Y^{(1)}]$.

$\mathcal{U}_\lambda = \Gamma(\mathcal{D}^{\lambda}_{G/B})$.

Thm: 1) $\mathcal{U}_{-\rho}$ is an Azumaya algebra over $Y^{(1)}$.

2) $\mathcal{D}^{-\rho}_{G/B} \xleftarrow{\sim} \pi^*\mathcal{U}_{-\rho}$.

Proof: (1): Can talk about the locus in $Y^{(1)}$, where $\mathcal{U}_{-\rho}$ is Azumaya. It's open. $G \curvearrowright \mathcal{U}_{-\rho}, Y^{(1)}$ in compatible way & Azumaya locus is G-stable. The closure of any G-orbit in $Y^{(1)}$ contains $\{0\}$ & generic rk of $\mathcal{U}_{-\rho}$ is $p^{\dim \mathcal{N}}$.

So we reduce to proving that $\mathcal{U}^{0}_{-\rho}$:= fiber of $\mathcal{U}_{-\rho}$ at 0 is matrix algebra of size $p^{\dim \mathcal{N}/2}$ ($\dim \mathcal{N}/2 = \dim G/B$)

Define a baby Verma module:

Consider Verma module $\Delta_{-\rho} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{F}_{-\rho}$, over $\mathcal{U}_{-\rho}$, free rk 1 module over $\mathcal{U}(\mathfrak{n}^-)$, so $\Delta_{-\rho}$ is a free rk $p^{\dim \mathfrak{n}_-}$-module over $S(\mathfrak{n}^{-(1)})$, this is central in $\mathcal{U}(\mathfrak{g})$, $\rightsquigarrow$ specialization of $\Delta_{-\rho}$ to $0 \in (\mathfrak{n}^{-(1)})^*$, called baby Verma, denoted by $\delta_{-\rho}$. It's a $\mathcal{U}^{0}_{-\rho}$-module of dimension $p^{\dim \mathfrak{n}_-} = p^{\dim \mathcal{N}/2}$.

Exercise: 1) $\delta_{-\rho}$ is irreducible over $\mathcal{U}^{0}_{-\rho}$.

2) it's the only irreducible

3) $\mathfrak{g}$-action on $\delta_{-\rho}$ integrates to G (Steinberg rep'n)

4*) $\delta_{-\rho}$ has no higher self-extensions.

$\Downarrow$

$\mathcal{U}^{0}_{-\rho} \xrightarrow{\sim} \mathrm{End}_{\mathbb{F}}(\delta_{-\rho})$, finishing (1).

(2): $\mathcal{U}_{-\rho} \longrightarrow \Gamma(\mathcal{D}^{-\rho}_{G/B})$, $\mathcal{O}_{X^{(1)}} \hookrightarrow \mathcal{D}^{-\rho}_{G/B}$ (center) & these homomorphisms coincide on $\mathbb{F}[Y^{(1)}]$ $\rightsquigarrow$ sheaf of algebra homom'm $\pi^* \mathcal{U}_{-\rho} \longrightarrow \mathcal{D}^{-\rho}_{G/B}$, homomorphism of Azumaya algebras of the same rank, so it's isomorphism $\square$

Recall for $y \in Y^{(1)} \rightsquigarrow Y^{(1)\wedge_y} := \operatorname{Spec} \mathbb{F}[Y^{(1)}]^{\wedge_y} \rightsquigarrow$
$X^{(1)\wedge_y} := Y^{(1)\wedge_y} \times_{Y^{(1)}} X^{(1)}$ – "small neigh'd of $\pi^{-1}(y)$"

Cor: $\forall\, \lambda \in \mathfrak{X}(B) \Rightarrow \mathcal{D}^{\lambda}_{G/B}|_{X^{(1)\wedge_y}}$ splits.

Proof: All $\mathcal{D}^{\lambda}_{G/B}$ are Morita equivalent, so we reduce to $\lambda = -\rho$.
$\mathcal{D}^{-\rho}_{G/B}|_{X^{(1)\wedge_y}} \simeq \pi^*(\underbrace{\mathcal{U}_{-\rho}|_{Y^{(1)\wedge_y}}}_{\text{splits}}) \Rightarrow \mathcal{D}^{-\rho}_{G/B}|_{X^{(1)\wedge_y}}$ splits

□

2) Applications to representation theory

$\lambda \in \mathfrak{h}^*_{\mathbb{F}_p}$, assume it's p-regular ($\langle \lambda+\rho, \alpha^\vee \rangle \neq 0\ \forall$ roots α).
BMR derived localization thm:

$$R\Gamma : D^b(\mathrm{Coh}\, \mathcal{D}^{\lambda}_{G/B}) \xrightarrow{\sim} D^b(\mathcal{U}_\lambda\text{-mod})$$

Let $\mathcal{E}^y$ denote a splitting bundle for $\mathcal{D}^{\lambda}_{G/B}|_{X^{(1)\wedge_y}}$. Then have the following equivalences:

$$\begin{tikzcd}
D^b(\mathrm{Coh}\, \mathcal{D}^{\lambda}_{G/B}|_{X^{(1)\wedge_y}}) \arrow[r, "\sim", "R\Gamma"'] & D^b(\mathcal{U}_\lambda^{\wedge_y}\text{-mod}) \\
D^b(\mathrm{Coh}(X^{(1)\wedge_y})) \arrow[u, "\mathcal{E}^y \otimes \bullet"', "\wr"] \arrow[ur, "\sim", "R\Gamma(\mathcal{E}^y \otimes \bullet)"']
\end{tikzcd} \qquad (1)$$

Consequences: $\mathcal{U}_\lambda\text{-mod}^y := \{$fin. dim. $\mathcal{U}_\lambda$-modules supp'd at $y\} \subset \mathcal{U}_\lambda^{\wedge_y}$-mod
$\mathrm{Coh}_{\pi^{-1}(y)}(X^{(1)}) = \{$coh sheaves supported on $\pi^{-1}(y)\} \subset \mathrm{Coh}(X^{(1)\wedge_y})$.

Observations: • $D^b(\mathcal{U}_\lambda\text{-mod}^y) \longrightarrow D^b(\mathcal{U}_\lambda^{\wedge y}\text{-mod})$

$$D^b(Coh_{\pi^{-1}(y)}(X^{(1)})) \longrightarrow D^b(Coh\, X^{(1)\wedge y})$$

are full embeddings.

• $K_0(\mathcal{U}_\lambda\text{-mod}^y) \xleftarrow{\sim} K_0(\mathcal{U}_\lambda^y\text{-mod})$,

$K_0(Coh_{\pi^{-1}(y)} X^{(1)}) \xleftarrow{\sim} K_0(Coh\, \pi^{-1}(y))$.

Corollary of (1) & Observation

(i) $D^b(\mathcal{U}_\lambda\text{-mod}^y) \xleftarrow[R\Gamma(\mathcal{E}^y \otimes \cdot)]{\sim} D^b(Coh_{\pi^{-1}(y)} X^{(1)})$

$\downarrow K_0$

(ii) $K_0(\mathcal{U}_\lambda^y\text{-mod}) \xleftarrow{\sim} \underbrace{K_0(Coh\, \pi^{-1}(y))}_{\text{computable!!!}}$

3) Properties of splitting bundle.

Take $\lambda = 0$.

$\mathcal{E} := \mathcal{E}^{\wedge_0}$ on $X^{(1)\wedge_0}$-neigh'd of $G^{(1)}/B^{(1)}$ in $T^*(G^{(1)}/B^{(1)})$

as any splitting bundle, $\mathcal{E}$ is defined up to a twist w. a line bundle: $Pic(X^{(1)\wedge_0}) \xrightarrow{\sim} Pic(G^{(1)}/B^{(1)}) \simeq \mathfrak{X}(B^{(1)})$.

As we've seen in the beginning, $Fr_* \mathcal{O}_{G/B}$ is splitting bundle for $\mathcal{D}_{G/B}|_{G^{(1)}/B^{(1)}}$. We get $\mathcal{E}|_{G^{(1)}/B^{(1)}} \xrightarrow{\sim} Fr_* \mathcal{O}_{G/B}(\mu)$, $\mu \in \mathfrak{X}(B^{(1)})$

Example: $G = SL_2$, $G/B = \mathbb{P}^1 \rightsquigarrow$ rk p vector bdle $Fr_* \mathcal{O}_{\mathbb{P}^1}$ on $\mathbb{P}^{1(1)}$. Any vector bundle on $\mathbb{P}^1 \simeq \bigoplus$ line bundles.

$$H^i(\mathbb{P}^{1(1)}, Fr_* \mathcal{O}_{\mathbb{P}^1}) = H^i(\mathbb{P}^1, \mathcal{O}) = \begin{cases} \mathbb{F}, & i=0 \\ 0, & i>0 \end{cases}$$

$$\Rightarrow Fr_* \mathcal{O}_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^{1(1)}} \oplus \mathcal{O}_{\mathbb{P}^{1(1)}}(-1)^{\oplus p-1}$$

Indecomposable summands of $\mathcal{E}$.

Prop'n: $\mathcal{E}$ has exactly $|W|$ pairwise non-isom'c direct summands

Proof: $R\Gamma(\mathcal{E}\otimes \bullet): D^b(Coh\, X^{(1)\wedge_0}) \xrightarrow{\sim} D^b(\mathcal{U}_0^{\wedge_0}\text{-mod})$

Let I_0 be the max. ideal of 0 in $S(\mathfrak{g}^{(1)})$. Then $\mathcal{U}_0^{\wedge_0} = \varprojlim_n \mathcal{U}_0 / \mathcal{U}_0 I_0^n$. Therefore indecomposable projective modules over $\mathcal{U}_0^{\wedge} \xleftrightarrow{\sim}$ irreducible modules over $\mathcal{U}_0^0$.

Recall (Sect. 2) $K_0(\mathcal{U}_0^0\text{-mod}) \xleftrightarrow{\sim} K_0(Coh\, G^{(1)}/B^{(1)})$ has rk equal to $|W|$. So we have $|W|$ irreps

$R\Gamma(\mathcal{E}\otimes\bullet): \mathcal{E}^* \longmapsto \mathcal{U}_0^{\wedge_0} \rightsquigarrow$ bijection between:

- indecomposable summands of $\mathcal{U}_0^{\wedge_0}$ = indecomp. projectives (up to iso).
- indec. summands of $\mathcal{E}^*$ (up to iso). $\square$

Rep. th'c interpr'n of rks/multiplicities of indecomp. summands.

$x \in G^{(1)}/B^{(1)} \rightsquigarrow \mathcal{E}_x \in \mathcal{U}_0\text{-mod}$ w. $S(\mathfrak{g}^{(1)})$ acting via eval'n at 0, i.e. $\mathcal{E}_x \in \mathcal{U}_0^0\text{-mod}$: $\mathcal{E}_x = R\Gamma(\mathcal{E}\otimes\mathbb{F}_x)$, where $\mathbb{F}_x$ is sky-scraper at 0.

Def'n: The class of a point module in $K_0(U_0^c\text{-mod})$ is $[\mathcal{E}_x]$ (independent of x)

Proof of Prop'n: $\{\text{indec. summands of } \mathcal{E}\} \xleftrightarrow{\sim} \{U_0^c\text{-irreps}\}$

$$\mathcal{E}_M \longleftarrow\!\shortmid M$$

Lemma: (i) $\operatorname{rk} \mathcal{E}_M$ = multiplicity of M in $[\mathcal{E}_x]$

(ii) the multiplicity of $\mathcal{E}_M$ in $\mathcal{E}$ = $\dim M$.

Proof: $\mathcal{E}_M \rightsquigarrow$ indecomp. projective $U_0^{\hat{\lambda}_0}$-module $P_M = R\Gamma(\mathcal{E}\otimes\mathcal{E}_M^*)$

$\mathcal{F} \in D^b(Coh_{\zeta^{(1)}/B^{(1)}} X^{(1)}) \rightsquigarrow N = R\Gamma(\mathcal{E}\otimes\mathcal{F}) \in D^b(U_0\text{-mod}^\circ)$

$R\Gamma(\mathcal{E}\otimes\cdot)$ is an equivalence $\Rightarrow$

$$\begin{array}{ccc} \operatorname{Hom}_{D^b(Coh X^{(1)})}(\mathcal{E}_M^*, \mathcal{F}) & \xrightarrow{\sim} & \operatorname{Hom}_{D^b(U_0\text{-mod})}(P_M, N) \\ \wr\wr & & \| \\ H^0(\mathcal{E}_M\otimes\mathcal{F}) & & \text{mult. space of } M \text{ in } H^0(N) \end{array}$$

For $\mathcal{F} = \mathbb{F}_x$: l.h.s $= \mathcal{E}_{M,x}$ $\qquad$ $\underbrace{H^0(N)}_{\mathcal{E}_x}$

$\Rightarrow \operatorname{rk} \mathcal{E}_M$ = mult. of M in $\mathcal{E}_x$

Exercise: prove (ii) $\square$

Identification of $[\mathcal{E}_x]$.

Verma $\Delta(0) := U(\mathfrak{g})\otimes_{U(\mathfrak{b})} \mathbb{F}_0 \rightsquigarrow$ baby $\delta_0 := \mathbb{F}_0 \otimes_{S(\mathfrak{n}^{-(1)})} \Delta(0)$.

$\dim \delta_0 = p^{\dim \mathfrak{n}^-} = \operatorname{rk} \mathcal{E}$.

Proposition: $[\mathcal{E}_x] = [\delta_0^*]$

Note: $\delta_0^* \in \mathcal{U}_0^0$-mod

Exercise: $[\delta_0] = [\delta_0^*]$

Proof of Prop'n: take $x = B^{(1)}/B^{(1)} \in G^{(1)}/B^{(1)}$, prove $\mathcal{E}_x \simeq \delta_0^*$.

$\mathcal{E}|_{G^{(1)}/B^{(1)}} \simeq Fr_* \mathcal{O}_{G/B} \Rightarrow \mathcal{E}_x := \mathbb{F}[Fr^{-1}(x)]$ ($Fr: G/B \to G^{(1)}/B^{(1)}$) – $\mathfrak{g}$-module, where $\mathfrak{g}$ acts by derivations.

Observations:

1) $1 \in \mathbb{F}[Fr^{-1}(x)]$, $\mathfrak{g}.1 = 0$.

2) $\xi \mapsto \xi_{G/B}: \mathfrak{n}^- \xrightarrow{\sim} T_{B/B} G/B$ (B/B is the point in $Fr^{-1}(x)$)

Exercise: every $\mathfrak{n}^-$-submodule in $\mathbb{F}[Fr^{-1}(x)]$ includes 1 (i.e. 1 co-generates the $\mathfrak{n}^-$-module $\mathbb{F}[Fr^{-1}(x)]$.

3) $T \curvearrowright \mathbb{F}[Fr^{-1}(x)]$, 1 has wt 0 & all other wt. vectors have positive wt.

Look at $\mathbb{F}[Fr^{-1}(x)]^*$. It has 1-dim'l wt 0 subspace that (by Exercise) generates $\mathbb{F}[Fr^{-1}(x)]^*$ & all other wts are negative $\Rightarrow$ wt 0 wt. space is killed by $\mathfrak{b}$.

By previous paragraph $\rightsquigarrow \Delta(0) \twoheadrightarrow \mathbb{F}[Fr^{-1}(x)]^*$; $S(\mathfrak{g}^{(1)})$ acts on $\mathbb{F}[Fr^{-1}(x)]^*$ via evaluation at 0 so the epimorphism factors through $\delta_0 \twoheadrightarrow \mathbb{F}[Fr^{-1}(x)]^*$

of same $\dim = p^{\dim G/B}$ $\Rightarrow \delta_0 \xrightarrow{\sim} \mathbb{F}[Fr^{-1}(x)]^*$ □

Example: $G = SL_2$: $i = 0, 1, .. p-1 \rightsquigarrow SL_2$-irrep $L(i)$ w. highest wt i. Still irreducible over $\mathfrak{g}$: $L(0) = \mathbb{F}$, $L(p-2) \in \mathcal{U}_0^0$-mod.

$\delta_0 \to \mathbb{F} = L(0)$, $\ker \delta_0 = L(p-2)$

Both simples occur in the class of a point module w. mult $= 1$

$\Longleftrightarrow$ all summands of $\mathcal{E}$ are line bundles?

$$\mathcal{E}|_{G^{(1)}/B^{(1)}} = \mathcal{O} \oplus \mathcal{O}(-1)^{\oplus p-1}$$

$$\mathcal{E} = \mathcal{O}_{X^{(1)}\lambda_0} \oplus \mathcal{O}_{X^{(1)}\lambda_0}(-1)^{\oplus (p-1)}.$$