Quantization in char p.
Lecture 7.

1) Hilbert schemes & Procesi bundles.

1.1) Varieties. $\mathfrak{g}=\mathfrak{gl}_n(\mathbb{C})\supset\mathfrak{h}\simeq\mathbb{C}^n$, Cartan (diag. matrices)
$W\curvearrowright\mathfrak{h}$: $V=\mathfrak{h}\oplus\mathfrak{h}^*\curvearrowleft W$ symplectic
$\rightsquigarrow Y:=V/W$, singular Poisson variety.
$T=(\mathbb{C}^\times)^2\curvearrowright V$: $(t_1,t_2).(v,\alpha)=(t_1^{-1}v, t_2^{-1}\alpha)$ descends to Y
Subtori $T_h=\{(t,t^{-1})\mid t\in\mathbb{C}^\times\}$ acts via Hamiltonian action
$T_c=\{(t,t)\mid t\in\mathbb{C}^\times\}$ has contracting action on V (& on Y).

Y has conical symplectic resolution $X=\operatorname{Hilb}_n(\mathbb{C}^2)$, parametrizing codim n ideals in $\mathbb{C}[x,y]$
- $T\curvearrowright X$ from $T\curvearrowright\mathbb{C}[x,y]$
- Have nat'l map $\rho: X\to Y=(\mathbb{C}^2)^n/S_n$ (Hilbert-Chow): ideal $\mapsto$ support counted w. multiplicities

T-equiv't, projective, an isomorphism over the locus of n pairwise distinct pts, so ρ is birational.
Since Y is normal, $\rho^*:\mathbb{C}[Y]\xrightarrow{\sim}\mathbb{C}[X]$.

1.2) Procesi bundle $H=\mathbb{C}[V]\#W$ $(=\mathbb{C}[V]\otimes\mathbb{C}W)$
- graded $\mathbb{C}[Y]=\mathbb{C}[V]^W$-algebra, actually bigraded w. $\mathfrak{h}^*$ in deg $(1,0)$, $\mathfrak{h}$ in deg $(0,1)$ and W in deg $(0,0)$

Let $\mathcal{P}$ be a vector bundle on X s.t. $\operatorname{End}(\mathcal{P})\xrightarrow{\sim}H$, an iso of $\mathbb{C}[Y]$-algebres $\rightsquigarrow$ $W \curvearrowright \mathcal{P}$ be vector bundle autom's.

Lemma: each fiber of $\mathcal{P}$ is $\mathbb{C}W$ as a W-module.

Proof: enough to show $\exists\, x \in X \mid \mathcal{P}_x \simeq_W \mathbb{C}W$ b/c X is connected.

$V^\circ = \{v = (p_1, \dots, p_n) \in V = (\mathbb{C}^2)^n \mid p_i \neq p_j \Leftarrow i \neq j\} = \{v \in V \mid W_v = \{1\}\}$

$\rho: X \to Y$ is an iso over V°/W

$\mathrm{End}(\mathcal{P}) \xrightarrow{\sim} H$ specialize to $x \in V^\circ/W \rightsquigarrow m_x \in \mathbb{C}[V]^\Gamma$ is max. ideal

$$\mathrm{End}(\mathcal{P}_x) \xrightarrow{\sim} H/(m_x) = [\mathbb{C}[V]/(m_x)] \# W$$

$\mathrm{End}(\mathcal{P}_x) \curvearrowright \mathcal{P}_x$, $\quad W \curvearrowright \mathbb{C}[V]/(m_x)$

$\mathbb{C}[V]/(m_x)$

$\cong$

$\mathbb{C}[\underbrace{\text{preimage of } x \text{ in } V^\circ}_{|W| \text{ distinct pts}}]$ – irred. module of dim $= |W|$, regular W-module

$\Rightarrow \mathcal{P}_x \simeq_W \mathbb{C}W$ $\square$

Corollary: usual and sign invariants $\mathcal{P}^{S_n}, \mathcal{P}^{sgn}$ are line bundles.

Fact: $\mathrm{Pic}(X) \xrightarrow{\sim} \mathbb{Z}$ w. $\mathcal{O}(1)$ being ample.

$\mathcal{O}(n) \longmapsfrom n$

Definition/Thm: A Procesi bundle on X is the unique T-equivariant vector bundle $\mathcal{P}$ w. T-equiv't $\mathbb{C}[Y]$-alg. isom'm $\mathrm{End}(\mathcal{P}) \xrightarrow{\sim} H$ s.t.

(i) $\mathrm{Ext}^i(\mathcal{P},\mathcal{P}) = \{0\}$ $\forall\, i>0$.
(ii) $\mathcal{P}^{S_n} \xrightarrow{\sim} \mathcal{O}_X$, T-equiv. isom'm
(iii) $\mathcal{P}^{sgn} \xrightarrow{\sim} \mathcal{O}(1)$.

Constructions: Haiman, Bezrukavnikov-Kaledin (via quantizations in char p – to be explained later), Ginzburg
Uniqueness – I.L.

Rem: Can generalize this to $Y = V/\Gamma$, $\Gamma \subset Sp(V)$ is finite subgroup s.t. Y admits a symplic resol'n & T_c-equiv. bundles $\mathcal{P}$ w. $\mathrm{End}(\mathcal{P}) = H := \mathbb{C}[V]\#\Gamma$ & (i), (ii) are classified by el'ts of Namikawa-Weyl gr'p of Y ($\mathbb{Z}/2\mathbb{Z}$ for $Y = (\mathfrak{h}\oplus\mathfrak{h}^*)/S_n$).

1.3) Connection to Combinatorics.

Fact: T_h-fixed points in $X = \mathrm{Hilb}_n(\mathbb{C}^2)$ are T-fixed $\xleftrightarrow{\sim}$ monomial ideals in $\mathbb{C}[x,y]$ $\xleftrightarrow{\sim}$ Young diagrams (λ Young diagram, fill it w. monomials $\begin{array}{|c|c|}\hline y^2 & \\ \hline y & \\ \hline 1 & x \\ \hline\end{array}$: take the ideal = span of remaining monomials).

$\lambda \rightsquigarrow x_\lambda \in X^T \rightsquigarrow$ fiber $\mathcal{P}_\lambda$ carries a T-action & commuting W-action, i.e. is bigraded W-module ($\simeq \mathbb{C}W$)
$\rightsquigarrow$ symmetric polynomial w. coeff's in $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ (Frobenius char.)

Thm (Macdonald positivity; Haiman): This polynomial is the modified Macdonald polynomial $\tilde{H}_\lambda$.

Alternative proof was found by Bezrukavnikov-Finkelberg (using quant'ns in char p). One advantage: generalizes to wreath-Macdonald polynomials (assoc. to $S_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n$)

Another application (of constr'n via quant'ns in char p): $\alpha \geq 0$
Boixeda Alvarez - I.L. $\quad H^i(\mathcal{P}^* \otimes \mathcal{P} \otimes \mathcal{O}(\alpha)) = \{0\}$ for $i > 0$
$\rightsquigarrow \Gamma(\mathcal{P}^* \otimes \mathcal{P} \otimes \mathcal{O}(\alpha)) = R\Gamma(\mathcal{P}^* \otimes \mathcal{P} \otimes \mathcal{O}(\alpha))$ – H-bimodule related to "∇^α".

2) Hamiltonian reduction

2.1) Setting:

$U := \mathbb{C}^n$, $G = GL(U) \curvearrowright U$, $R := End(U) \oplus U \curvearrowleft G$
$\xi \mapsto \xi_R : \mathfrak{g} \longrightarrow Vect(R)$, $\xi_R = ([\xi, \cdot], \xi)$

$$T^*R = R \oplus R^* = [\,End(U)^* \simeq End(U) \text{ via tr-pairing}\,] = \underset{(A,B,i,j)}{\overset{\uplus}{End(U)^{\oplus 2} \oplus U \oplus U^*}}.$$
$\circlearrowleft$
G

Moment map $\mu : T^*R \longrightarrow \mathfrak{g}^* \underset{tr}{\simeq} \mathfrak{g}$
$\langle \mu(A,B,i,j), \xi \rangle = \langle \xi_R(A,i), (B,j) \rangle = \langle [\xi, A], B \rangle + \langle \xi i, j \rangle =$
$= tr\left(([A,B] + ij)\xi\right)$ so $\mu(A,B,i,j) = [A,B] + ij \in End(U) = \mathfrak{g}$.

2.2) $Y = V/W$ as Hamiltonian reduction

Thm (Gan-Ginzburg: "Almost commuting...")

$Y \xrightarrow{\sim} \mu^{-1}(0)/\!/G$ (Poisson & T-equiv't, $T \curvearrowright R \oplus R^*$ similarly to before)

Sketch of proof:

Step 1: GG proved: $\mu^{-1}(0)$ is the union of $n+1$ irreducible components of codim $= n^2 \Longrightarrow \mu^{-1}(0)$ is a complete intersection. Each component has a free G-orbit so μ is a submersion on this orbit. So $\mu^{-1}(0)$ is generically reduced, hence [it's complete intersection] reduced. Hence $\mu^{-1}(0)/\!/G$ is reduced.

Step 2: produce $Y \to \mu^{-1}(0)/\!/G$: Have $V \hookrightarrow \mu^{-1}(0)$:
$((x_1, \dots x_n), (y_1 \dots y_n)) \mapsto (\mathrm{diag}(x_1, \dots x_n), \mathrm{diag}(y_1, \dots y_n), 0, 0)$.
images of S_n-conjugate pts are conjugate via monomial matrices, hence lie in the same G-orbit $\rightsquigarrow$

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, two heads] & \mu^{-1}(0) \arrow[d, two heads] \\ V/W \arrow[r, "\iota"] & \mu^{-1}(0)/\!/G \end{tikzcd}$$

Step 3: ι is a closed embedding. Result (of Weyl) says that $\mathbb{C}[V]^W$ is generated by $\sum_{i=1}^n x_i^k y_i^\ell$ ($k, \ell \geq 0$). Consider $F_{k,\ell} \in \mathbb{C}[\mu^{-1}(0)]^G$, $F_{k,\ell}(A, B, i, j) = tr(A^k B^\ell) \Rightarrow \iota^* F_{k,\ell} = \sum x_i^k y_i^\ell$
So ι^* is surjective.

Step 4: Since ι is a closed embedding into a reduced scheme so to prove it's iso $\Leftrightarrow$ it's surjective; pts of $\mu^{-1}(0)/\!/G \xleftrightarrow{\sim}$ closed G-orbits in $\mu^{-1}(0) = \{(A,B,i,j) \mid \underbrace{[A,B]}_{} + \underbrace{ij}_{} = 0\}$ $rk \leq 1$

Fact: $rk\,[A,B] \leq 1 \Rightarrow \exists$ basis where the operators A,B are upper triangular

Exercise: Show that every closed G-orbit in $\mu^{-1}(0)$ intersects the locus $\{diag(x_1,\dots x_n), diag(y_1,\dots y_n), 0, 0\}$

$\Rightarrow \iota$ is surjective. $\square$

2.3) $Hilb_n(\mathbb{C}^2)$ as GIT Hamiltonian reduction

for $G \curvearrowright \mu^{-1}(0)$ w.r.t. certain character.

Basics on GIT quotient: G reductive group$/\mathbb{C}$, $G \curvearrowright Z$ - affine variety (or finite type scheme), $\theta: G \to \mathbb{C}^\times$.

$\rightsquigarrow$ graded algebra $\bigoplus_{n\geq 0} \underbrace{\mathbb{C}[Z]^{G,n\theta}}_{} \leftarrow$ semiinvariants.

$\rightsquigarrow$ GIT quotient $Z/\!/^\theta G := Proj(\bigoplus_{n \geq 0} \mathbb{C}[Z]^{G,n\theta})$

- glued from open affine charts of the following form:

$f \in \mathbb{C}[Z]^{G,n\theta}$, $n>0 \rightsquigarrow Z_f := \{z \in Z \mid f(z) \neq 0\} \curvearrowleft G$ (affine) $\rightsquigarrow Z_f/\!/G = \operatorname{Spec} \mathbb{C}[Z_f]^G$

$Z/\!/^\theta G$ is glued from $Z_f/\!/G$ ($Z_f/\!/G$ & $Z_{f'}/\!/G$ are glued along their common open subset $Z_{ff'}/\!/G$).

$Z/\!/^\theta G$ parametrizes closed orbits in "θ-semistable locus"

$Z^{\theta\text{-}ss} = \{ z \in Z \mid \exists\, n>0 \ \&\ f \in \mathbb{C}[Z]^{G, n\theta} \text{ s.t. } f(z) \neq 0 \}$

Example: $Z = T^*R$, $\theta = \det^{-1}$

Classical invariant theory shows that $\mathbb{C}[Z]^G$-algebra $\bigoplus_{n \geq 0} \mathbb{C}[Z]^{G, n\theta}$ is generated by elements of the form

$$\det(f_1(A,B)i, f_2(A,B)i, \dots, f_n(A,B)i), \quad f_1, \dots f_n \in \mathbb{C}\langle x, y \rangle$$

$(T^*R)^{\theta\text{-}ss} = \{ (A,B,i,j) \mid$ one of det's is nonzero $\Leftrightarrow U = \mathrm{Span}(f_1(A,B)i, \dots f_n(A,B)i) \Leftrightarrow \mathbb{C}\langle A,B\rangle i = U \}$

G-action on this locus is free. So $\mu^{-1}(0)/\!/^\theta G$ parameterizes all orbits in $\mu^{-1}(0)^{\theta\text{-}ss}$ & is smooth & symplectic.

Identification w. Hilbert scheme: $(\theta = \det^{-1})$

Exercise: $(A,B,i,j) \in \mu^{-1}(0)^{\theta\text{-}ss} \Rightarrow j = 0$, $[A,B] = 0$ & $\mathbb{C}[A,B]i = U$.

Identification

$$\begin{array}{ccc} \mu^{-1}(0)^{\theta\text{-}ss}/G & \longrightarrow & \mathrm{Hilb}_n(\mathbb{C}^2) \\ \Psi & & \Psi \\ (A,B,i) & \longmapsto & \{ f \in \mathbb{C}[x,y] \mid f(A,B) = 0 \Leftrightarrow f(A,B)i = 0 \} \end{array}$$

Exercise: show this is a bijection of sets.

Rems: • Have comm've diagram

$$\begin{tikzcd} Z^{\theta\text{-}ss} \arrow[r, hook] \arrow[d] & Z \arrow[d] \\ Z/\!/^\theta G \arrow[r, "\rho"] & Z/\!/G \end{tikzcd}$$

– quotient morphisms

where ρ: $\mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathbb{C}[Z]^{G, n\theta}\right) \longrightarrow \mathrm{Spec}\, \mathbb{C}[Z]^G$ - natural projective morphism.

If $Z = \mu^{-1}(0)$, then ρ is the Hilbert-Chow morphism $\mathrm{Hilb}_n(\mathbb{C}^2) \longrightarrow (\mathbb{C}^2)^n / S_n$

- Ample generator $\mathcal{O}(1)$ on X is the line bundle obtained from char'r $\det^{-1}$: $G \to \mathbb{C}^\times$ by equiv't descent for $\mu^{-1}(0)^{\theta\text{-}ss} \twoheadrightarrow \mu^{-1}(0)/\!/^{\theta} G$.