Quantizations in char p, Lecture 8.

Topic: quantizations of symmetric powers & Hilbert schemes

Goal: construct a filtered Frobenius constant quantization of $\mathrm{Hilb}_n(\mathbb{F}^2)$, $\mathbb{F}=\overline{\mathbb{F}}$, char $\mathbb{F}=p\gg 0$ w. global sections $\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{S_n}$.
Later we'll use this to construct Procesi bundle via splitting bundle construction

1) Quantizations of symmetric powers.

Setting: $\mathfrak{h}=\mathbb{C}^n$, $W=S_n \curvearrowright \mathfrak{h}$, $V=\mathfrak{h}\oplus\mathfrak{h}^* \rightsquigarrow Y=V/S_n$

$U=\mathbb{C}^n \curvearrowleft G=GL(U) \rightsquigarrow G \curvearrowright R:=\mathrm{End}(U)\oplus U \rightsquigarrow$

$T^*R=\mathrm{End}(U)^{\oplus 2}\oplus U\oplus U^*$

$\mu: T^*R \to \mathfrak{g}^*\simeq \mathrm{End}(U)$: $\mu(A,B,i,j)=[A,B]+ij$

Have seen: $V \longrightarrow \mu^{-1}(0)$, $(x_1,\dots x_n,y_1,\dots y_n)\mapsto$
$(\mathrm{diag}(x_1,\dots x_n), \mathrm{diag}(y_1,\dots y_n),0,0)$

$\rightsquigarrow V/S_n \xrightarrow{\sim} \mu^{-1}(0)/\!/G$.

Goal: construct filtered quant'ns of $\mathbb{C}[Y]=\mathbb{C}[V]^{S_n}$

- Easy: $\mathcal{D}(\mathfrak{h})^{S_n}$.
- Hamiltonian reduction: $\lambda\in\mathbb{C}\rightsquigarrow$

$\mathcal{D}(R)/\!/\!/_\lambda G=\left(\mathcal{D}(R)/\mathcal{D}(R)\{\xi_R-\lambda\,tr(\xi)\}\right)^G$ is a filt. quant'n of $\mathbb{C}[\mu^{-1}(0)]^G$ b/c μ^* sends a basis of $\mathfrak{g}$ to a reg. sequence in $\mathbb{C}[T^*R]$ ($\Leftrightarrow \mu^{-1}(0)$ is a complete intersection in T^*R) $\Rightarrow$
gr $\mathcal{D}(R)/\mathcal{D}(R)\{\xi_R-\lambda\,tr(\xi)\}\xrightarrow{\sim}\mathbb{C}[\mu^{-1}(0)]$

1.1) Main result.

Thm: $\mathcal{D}(R)///_0 G \xrightarrow{\sim} \mathcal{D}(\mathfrak{h})^{S_n}$, iso of filt. quant'n of $\mathbb{C}[V]^{S_n}$.

Sketch of proof: In Lec 1, we've seen $\mathcal{D}(\mathfrak{g})///_0 G \xrightarrow{\sim} \mathcal{D}(\mathfrak{h})^{S_n}$.

i) Construction of homomorphism:

$\mathcal{D}(R)///_0 G \curvearrowleft \mathbb{C}[R]^G = [\mathbb{C}^\times = \text{center of } G \text{ acts by } 0 \text{ on } \mathfrak{g} \text{ \& scaling on } U] = \mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{h}]^W \curvearrowright \mathcal{D}(\mathfrak{h})^W$.

Observe that have alg. homom's $\mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathcal{D}(R)///_0 G$ & $\mathbb{C}[\mathfrak{h}]^W \hookrightarrow \mathcal{D}(\mathfrak{h})^W$. Via any of these $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{h}]^W$ acts on it itself by multiplications.

Invariant orthog. form on $\mathfrak{g} \rightsquigarrow \Delta_{\mathfrak{g}} \in S(\mathfrak{g})^G \longrightarrow \mathcal{D}(R)///_0 G$

— · — · — · — · — · — $\rightsquigarrow \Delta_{\mathfrak{h}} \in S(\mathfrak{h})^W \longrightarrow \mathcal{D}(\mathfrak{h})^W$

$\delta^{-1} \circ \Delta_{\mathfrak{g}} \circ \delta$ acts on $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{h}]^W$ as $\Delta_{\mathfrak{h}}$, δ is Vandermonde.

Claim: the actions of $\mathcal{D}(R)///_0 G$ & $\mathcal{D}(\mathfrak{h})^W$ on $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{h}]^W$ by the same operators after conjugating the latter by δ.

Exercise: Show that $\mathbb{C}[\mathfrak{h}]^W$ & $\Delta_{\mathfrak{h}}$ generate $\mathbb{C}[V]^{S_n}$ as a Poisson algebra (hint: use Weyl's thm on generators of the latter)

Consequence: $\operatorname{gr} \mathcal{D}(R)///_0 G$, $\operatorname{gr} \mathcal{D}(\mathfrak{h})^W \xrightarrow{\sim} \mathbb{C}[V]^{S_n}$ as graded Poisson algebras. Exercise $\Rightarrow$ $\mathbb{C}[\mathfrak{h}]^W, \Delta_{\mathfrak{h}}$ generate $\mathbb{C}[V]^{S_n}$ & $\mathbb{C}[\mathfrak{g}]^G, \Delta_{\mathfrak{g}}$ generate $\mathcal{D}(R)///_0 G$.

Exercise: $\mathcal{D}(\mathfrak{h})^{S_n} \curvearrowright \mathbb{C}[\mathfrak{h}]^{S_n}$ is faithful
$\rightsquigarrow \mathcal{D}(R)/\!/\!/_0 G \longrightarrow \mathcal{D}(\mathfrak{h})^{S_n}$ (b/c algebras act by the same operators).

ii) Need to show $\mathcal{D}(R)/\!/\!/_0 G \longrightarrow \mathcal{D}(\mathfrak{h})^W$ is a filtered algebra isomorphism. One can check that this preserves filtration & on gr's, it's pullback homom'm $\mathbb{C}[\mu^{-1}(0)]^G \longrightarrow \mathbb{C}[V]^{S_n}$. The latter is an isomorphism, so $\mathcal{D}(R)/\!/\!/_0 G \xrightarrow{\sim} \mathcal{D}(\mathfrak{h})^W$ $\square$

1.2) Remarks:

1*) One can ask to generalize the theorem to arbitrary λ:
$\mathcal{D}(R)/\!/\!/_\lambda G \xrightarrow{\sim} eH_\lambda e$, spherical rational Cherednik algebra.

2) Have char $p \gg 0$ version of Thm: $\mathbb{F} = \overline{\mathbb{F}}$, char $\mathbb{F} = p \gg 0 \rightsquigarrow$
(I) $\mathcal{D}(R_{\mathbb{F}})/\!/\!/_0 G_{\mathbb{F}}$ is quant'n of $\mathbb{F}[V]^{S_n}$
(II) $\mathcal{D}(R_{\mathbb{F}})/\!/\!/_0 G_{\mathbb{F}} \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{S_n}$.
Check: reduction from char 0:
(III) $\mathbb{F}[\mu^{-1}(0)]^{G_{\mathbb{F}}} \xrightarrow{\sim} \mathbb{F}[V]^{S_n}$.

- $\mathbb{C} \rightsquigarrow \mathbb{Q}$: possible b/c everything is defined over $\mathbb{Q}$, so analogs of (I)-(III) hold.

- $\mathbb{Q} \rightsquigarrow$ finite localization of $\mathbb{Z}$: possible b/c all algebras in question are finitely generated: S = a finite loc'n of $\mathbb{Z}$.

(II) becomes $\mathcal{D}(R_S)/\!/\!/_0 G_S \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}_S)^{S_n}$.

· $S \rightsquigarrow \mathbb{F}$, $\mathbb{F}$ is an S-algebra so can $\mathbb{F}\otimes_S \bullet$.

$\mathbb{F}[V]^{S_n} \xleftarrow{\sim} \mathbb{F}\otimes_S S[V]^{S_n}$; $\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{S_n} \xleftarrow{\sim} \mathbb{F}\otimes_S \mathcal{D}(\mathfrak{h}_S)^{S_n}$

$\mathbb{F}[\mu^{-1}(0)]^{G_{\mathbb{F}}} \hookleftarrow \mathbb{F}\otimes_S S[\mu^{-1}(0)]^{G_S}$.

invariants mod p reductions mod p of invariants

It's an isomorphism:

$\mathbb{F}[V]^{S_n} \xrightarrow{\sim} \mathbb{F}\otimes_S S[V]^{S_n} \xrightarrow{\sim} \mathbb{F}\otimes_S S[\mu^{-1}(0)]^{G_S} \hookrightarrow \mathbb{F}[\mu^{-1}(0)]^{G_{\mathbb{F}}}$.

$V_{\mathbb{F}} \hookrightarrow \mu_{\mathbb{F}}^{-1}(0) \rightsquigarrow \mathbb{F}[\mu^{-1}(0)]^{G_{\mathbb{F}}} \longrightarrow \mathbb{F}[V]^{S_n}$, every closed G-orbit in $\mu^{-1}(0)$ intersects the image of V, exercise, the latter homom'm is injective. But all our homomorphisms are bigraded. The bigraded comp's are finite dimensional so the existence of a pair of monomorphisms implies both of them are isom'ms.

This establishes (III) (over $\mathbb{F}$)

Exercise: prove (I) & (II) (note $\operatorname{gr}[\mathcal{D}(R_{\mathbb{F}})/\mathcal{D}(R_{\mathbb{F}})\{\xi_R\}] \xrightarrow{\sim} \mathbb{F}[\mu^{-1}(0)]$).

2) Quantizations of Hilbert schemes.

2.1) Preliminaries.

$X = \mu^{-1}(0)/\!/^{\theta} G$, $\theta = \det^{-1}$

$\mu^{-1}(0)^{\theta\text{-}ss} = \{(A,B,i,0) \mid [A,B]=0, \mathbb{C}[A,B]i = U\}$ - principal G-bundle over X.

Char $p \gg 0$ story: $\mu_{\mathbb{F}}^{-1}(0)^{\theta\text{-}ss} = \{(A,B,i,0) \mid [A,B]=0, \mathbb{F}[A,B]i = U_{\mathbb{F}}\}$

$\supset$: same argument as in Lec 7.

$\subset$: easy part of Hilbert-Mumford criterion.

$\mu_{\mathbb{F}}^{-1}(0)^{\theta\text{-}ss} \longrightarrow X_{\mathbb{F}}$, a principal $G_{\mathbb{F}}$-bundle.

We'll define a quantization of $X_{\mathbb{F}}$ by quantum Hamiltonian reduction. Recall commut. diagram from Lec 5:

$$(1)\quad \begin{array}{ccc} S(\mathfrak{g}_{\mathbb{F}}^{(1)}) & \xrightarrow{\varphi} & \mathbb{F}[T^*R^{(1)}] \\ \downarrow & & \downarrow \\ \mathcal{U}(\mathfrak{g}_{\mathbb{F}}) & \xrightarrow{\varphi} & \mathcal{D}(R_{\mathbb{F}}) \end{array} \qquad \begin{array}{l} \varphi(\xi)=\xi_{R^{(1)}}. \\ \\ \\ \varphi(\xi)=\xi_R^p-\xi_R^{[p]}. \end{array}$$

2.2) Construction of reduction:

View $\mathcal{D}(R_{\mathbb{F}})$ as a sheaf on $T^*R^{(1)} \rightsquigarrow \mathcal{D}(R_{\mathbb{F}})^{\theta\text{-}ss}$: restr'n of $\mathcal{D}(R_{\mathbb{F}})$ to $T^*R^{(1)\theta\text{-}ss} \rightsquigarrow \mathcal{D}(R_{\mathbb{F}})|_{(\mu^{(1)})^{-1}(0)^{\theta\text{-}ss}}$ - coherent sheaf of algebras on $(\mu^{(1)})^{-1}(0)^{\theta\text{-}ss}$.

$G_{\mathbb{F}} \curvearrowright \mathcal{D}(R_{\mathbb{F}})|_{(\mu^{(1)})^{-1}(0)^{\theta\text{-}ss}}$

consider $G_1 := \ker[G_{\mathbb{F}} \longrightarrow G_{\mathbb{F}}^{(1)}]$ - finite group scheme w. single pt.

$G_1 \curvearrowright \mathcal{D}(R_{\mathbb{F}})|_{(\mu^{(1)})^{-1}(0)^{\theta\text{-}ss}}$ by $\mathcal{O}$-linear automorphisms.

$\mathfrak{g} \longrightarrow \mathcal{D}(R_{\mathbb{F}})|_{(\mu^{(1)})^{-1}(0)^{\theta\text{-}ss}}$ is quantum comoment map for G_1-action $\rightsquigarrow$ coherent sheaf of $\mathcal{O}_{(\mu^{(1)})^{-1}(0)^{\theta\text{-}ss}}$-algebras

$$\left(\mathcal{D}(R_{\mathbb{F}})|_{(\mu^{(1)})^{-1}(0)^{\theta\text{-}ss}}\right)///_0\, G_1 = \left(\mathcal{D}(R)^{\theta\text{-}ss}/\mathcal{D}(R)^{\theta\text{-}ss}\{\xi_R\}\right)^{G_1},$$

$G_{\mathbb{F}}^{(1)}$-equivariant; $(\mu^{(1)})^{-1}(0)^{\theta\text{-}ss} \longrightarrow X_{\mathbb{F}}^{(1)}$, principal $G_{\mathbb{F}}^{(1)}$-bundle

By definition, $\mathcal{D}(R)^{\theta\text{-}ss}///_0\, G$ is the $G_{\mathbb{F}}^{(1)}$-equiv. descent of

$\left(\mathcal{D}(R)^{\theta\text{-}ss}/\mathcal{D}(R)^{\theta\text{-}ss}\{\xi_R\}\right)^{G_1}$, sheaf of $\mathcal{O}_{X^{(1)}_{\mathbb{F}}}$-algebras.

2.3) Frobenius constant quantization:

Claim: $\mathcal{D}(R)^{\theta\text{-}ss}/\!/\!/_0 G$ is a filtered Frobenius constant quantization of $X_{\mathbb{F}}$.

Proof: Need to show:

(i) $\mathcal{D}(R)^{\theta\text{-}ss}/\!/\!/_0 G$ is a filtered quantization

(ii) $\operatorname{gr} \mathcal{D}(R)^{\theta\text{-}ss}/\!/\!/_0 G \longrightarrow \mathcal{O}_{X_{\mathbb{F}}}$ intertwines embeddings from $\mathcal{O}_{X^{(1)}_{\mathbb{F}}}$.

Notation: $\hat{\mathcal{R}}_\hbar$ - completed Rees constr'n

$$\hat{\mathcal{R}}_\hbar\left(\mathcal{D}(R)^{\theta\text{-}ss}/\!/\!/_0 G\right) \xrightarrow{\sim} \left[\hat{\mathcal{R}}_\hbar \mathcal{D}(R)\right]^{\theta\text{-}ss}/\!/\!/_0 G$$

want to check: formal quantization of $X_{\mathbb{F}}$.

Recall (bonus to Lec 1) if $\mathcal{U}$ is an affine sympl'c variety w. Hamiltonian G-action s.t. $\mu^{-1}(0)$ is a princ. G-bundle over $\mu^{-1}(0)/G$ & $\mathcal{A}_\hbar$ is a formal quant'n of $\mathbb{F}[\mathcal{U}]$, w. Hamilt. G-action, then $\mathcal{A}_\hbar/\!/\!/_0 G$ is a formal quant'n of $\mathbb{F}[\mu^{-1}(0)]^G$.

Exercise:
- Deduce (i) from this reminder.
- Deduce (ii) from (1).

2.4) Identification of global sections.

Prop'n: Have a filtered algebra isomorphism

$$\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{S_n} = \mathcal{D}(R_{\mathbb{F}})/\!/\!/_0 G_{\mathbb{F}} \xrightarrow{\sim} \Gamma\left(\mathcal{D}(R_{\mathbb{F}})^{\theta\text{-}ss}/\!/\!/_0 G\right)$$

Proof: Step 1: construct a homomorphism of filtered algebras.
$\mathcal{D}(R_{\mathbb{F}})^{\theta\text{-ss}} /\!/\!/_0 G_{\mathbb{F}}$ obtained from gluing the algebras
$\mathcal{D}(R_{\mathbb{F}})|_{U} /\!/\!/_0 G_{\mathbb{F}}$, where U is a $G_{\mathbb{F}}$-stable open affine subvariety in $(T^*R^{(1)})^{\theta\text{-ss}}$. Notice that Hamiltonian reduction is functorial:
$\mathcal{D}(R_{\mathbb{F}}) \longrightarrow \mathcal{D}(R_{\mathbb{F}})|_{U}$ (restriction), $G_{\mathbb{F}}$-equivariant & intertwines quantum comoment maps $\rightsquigarrow \mathcal{D}(R_{\mathbb{F}}) /\!/\!/_0 G \longrightarrow \mathcal{D}(R_{\mathbb{F}})|_{U} /\!/\!/_0 G_{\mathbb{F}}$
$\rightsquigarrow \mathcal{D}(R_{\mathbb{F}}) /\!/\!/_0 G \xrightarrow[\psi]{} \Gamma(\mathcal{D}(R_{\mathbb{F}})^{\theta\text{-ss}} /\!/\!/_0 G_{\mathbb{F}})$, of filt. algebras.

Step 2: Show that it's an isomorphism. Similarly, have a graded algebra homom'm $\mathbb{F}[\mu_{\mathbb{F}}^{-1}(0)]^{G_{\mathbb{F}}} \xrightarrow[\psi_0]{} \mathbb{F}[\mu_{\mathbb{F}}^{-1}(0) /\!/^{\theta} G_{\mathbb{F}}]$, it's ρ^* where $\rho: \mu_{\mathbb{F}}^{-1}(0) /\!/^{\theta} G_{\mathbb{F}} \longrightarrow \mu_{\mathbb{F}}^{-1}(0) /\!/ G_{\mathbb{F}}$. In our case ρ is an isomorphism.

Compatibility: have commutative diagram

$$\begin{tikzcd}
\operatorname{gr} \mathcal{D}(R_{\mathbb{F}}) /\!/\!/_0 G_{\mathbb{F}} \arrow[r, "\operatorname{gr}\psi"] \arrow[d, "\wr"'] & \operatorname{gr} \Gamma(\mathcal{D}(R_{\mathbb{F}})^{\theta\text{-ss}} /\!/\!/_0 G_{\mathbb{F}}) \arrow[d, hook] \\
\mathbb{F}[Y] \arrow[r, "\sim", "\psi_0"'] & \mathbb{F}[X]
\end{tikzcd}$$

$\Rightarrow \operatorname{gr}\psi$ is an isomorphism $\Rightarrow$ ψ is isomorphism $\square$