Quantizations in char p, lecture 9.
Construction of Procesi bundle.

0) Recap We constructed a filtered Frobenius constant quantization $\mathcal{D}$ of $X_{\mathbb{F}} = Hilb_n(\mathbb{F}^2)$, $\mathcal{D} := D(R_{\mathbb{F}})/\!\!/\!\!/_0^\theta G_{\mathbb{F}}$

Construction: based on commut. diagram

$$\begin{array}{ccc} S(\mathfrak{g}_{\mathbb{F}}^{(1)}) & \longrightarrow & \mathbb{F}[T^*R^{(1)}] \\ \downarrow & & \downarrow \\ U(\mathfrak{g}_{\mathbb{F}}) & \longrightarrow & D(R_{\mathbb{F}}) \end{array}$$

$$D(R_{\mathbb{F}})^{\theta\text{-}ss} \rightsquigarrow D(R)|_{(\mu^{(1)})^{-1}(0)^{\theta\text{-}ss}} \rightsquigarrow D(R)|_{(\mu^{(1)})^{-1}(0)^{\theta\text{-}ss}}/\!\!/\!\!/_0 G_1 \qquad (1)$$
$$\rightsquigarrow D(R)/\!\!/\!\!/_0^\theta G := \left(D(R)|_{(\mu^{(1)})^{-1}(0)^{\theta\text{-}ss}}/\!\!/\!\!/_0 G_1\right)^{G^{(1)}}.$$

Rem: 1) 2nd & 3rd sheaves are Morita equivalent (via bimodule $D(R_{\mathbb{F}})^{\theta\text{-}ss}/D(R_{\mathbb{F}})^{\theta\text{-}ss}\{\xi_R\}$) Azumaya algebras on $(\mu^{(1)})^{-1}(0)^{\theta\text{-}ss}$ (Bezrukavnikov-Finkelberg-Ginzburg).

2) Can also construct $D(R_{\mathbb{F}})/\!\!/\!\!/_0 G_{\mathbb{F}}$ by similar producedure, in 3 steps

$$\mathcal{D}(R_{\mathbb{F}}) \rightsquigarrow \mathcal{D}(R)|_{(\mu^{(1)})^{-1}(0)} \rightsquigarrow \mathcal{D}(R)|_{(\mu^{(1)})^{-1}(0)} /\!/\!/_0 G_1$$
$$\rightsquigarrow \mathcal{D}(R)/\!/\!/_0^\theta G := \left(\mathcal{D}(R)|_{(\mu^{(1)})^{-1}(0)} /\!/\!/_0 G_1\right)^{G^{(1)}}. \qquad (2)$$

On each step, each of the algebras in (2) has homomorphism to global sections of the corresponding sheaf in (1), linear w.r.t. the algebra of function. In particular,

$$\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{S_n} = \mathcal{D}(R)/\!/\!/_0 G_{\mathbb{F}} \xrightarrow{\sim} \Gamma(\mathcal{D}(R)/\!/\!/_0^\theta G_{\mathbb{F}}) \text{ is linear}$$

over $\mathbb{F}[(\mu^{(1)})^{-1}(0)]^{G^{(1)}_{\mathbb{F}}} = \mathbb{F}[Y^{(1)}_{\mathbb{F}}]$

1) Roadmap

We construct Procesi bundle $\mathcal{P}$ on X (over $\mathbb{C}$) in 4 steps:

1) The restriction $\mathcal{D}^{\wedge_0}$ of $\mathcal{D}$ to $X^{(1)\wedge_0}_{\mathbb{F}} := \operatorname{Spec} \mathbb{F}[Y^{(1)}]^{\wedge_0} \times_{Y^{(1)}_{\mathbb{F}}} X^{(1)}_{\mathbb{F}}$ splits, let $\mathcal{E}$ be a splitting bundle.

2) $\exists$ $k>0$, idempotent $\varepsilon \in Mat_k(\Gamma(\mathcal{D}^{\wedge_0})) = Mat_k(\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{\wedge_0 S_n})$ s.t. $\varepsilon Mat_k(\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{\wedge_0 S_n})\varepsilon \xrightarrow{\sim} \mathbb{F}[V^{(1)}]^{\wedge_0} \# S_n$.

$\rightsquigarrow \mathcal{P}'_{\mathbb{F}} = \varepsilon(\mathcal{E}^{\oplus k})$: $Ext^i(\mathcal{P}'_{\mathbb{F}}, \mathcal{P}'_{\mathbb{F}}) = 0$ $\forall i>0$, $End(\mathcal{P}'_{\mathbb{F}}) \xrightarrow{\sim} \mathbb{F}[V^{(1)}]^{\wedge_0} \# S_n$.

3) By Lec 4, $\mathcal{P}'_{\mathbb{F}}$ has $\mathbb{G}_m$-equiv't structure. Can modify this s.t. $End(\mathcal{P}'_{\mathbb{F}}) \simeq \mathbb{F}[V^{(1)}]^{\wedge_0} \# S_n$ can be made $\mathbb{G}_m$-equiv't. The resulting $\mathbb{G}_m$-equivariant bundle, $\mathcal{P}_{\mathbb{F}}$, on $X^{(1)}_{\mathbb{F}}$ satisfies:

$Ext^i(\mathcal{P}_{\mathbb{F}}, \mathcal{P}_{\mathbb{F}}) = 0$ $\forall i>0$, $End(\mathcal{P}_{\mathbb{F}}) \xrightarrow{\sim} \mathbb{F}[V^{(1)}] \# S_n$.

4) View $\mathcal{P}_{\mathbb{F}}$ as a bundle on $X_{\mathbb{F}} \simeq X_{\mathbb{F}}^{(1)}$. We can lift $\mathcal{P}_{\mathbb{F}}$ to char 0 (Lec 4): $\mathbb{F} \rightsquigarrow \mathbb{F}_q \rightsquigarrow \underbrace{S^{\wedge q}}_{\text{ring of "p-adic integers"}} \rightsquigarrow \overline{\mathrm{Frac}(S^{\wedge q})} \simeq \mathbb{C}$

Need to show that the deformation to $S^{\wedge q}$, $\mathcal{P}_{S^{\wedge q}}$, satisfies
$\mathrm{End}(\mathcal{P}_{S^{\wedge q}}) \xrightarrow{\sim} S^{\wedge q}[V] \# S_n$.

We'll discuss 1) & 2) in this lecture & leave 4) (+ Macdonald positivity) for Lec 10 = the last lecture.

2) Splitting. Let $\rho: X \longrightarrow Y$ is resol'n of singularities.

Prop'n (Bezrukavnikov-Kaledin): $\exists$ Azumaya algebra, $\mathcal{A}$, on $Y_{\mathbb{F}}^{(1)}$ s.t. $\mathcal{D}$ & $\rho^*\mathcal{A}$ are Morita equivalent.

Cor: $\mathcal{D}^{\wedge_0}$ splits.

Rem: Prop'n is similar to the case of $T^*(G/B)$ (analog of $\mathcal{A}$ was $\mathcal{U}_{-\rho}$).

Def'n (Brauer group): Z is a scheme, the Brauer group $Br(Z)$ consists of Azumaya algebras up to Morita equivalence ($\mathcal{A} \sim \mathcal{B}$ if $\mathcal{A} \otimes \mathcal{B}^{opp}$ splits) w. addition induced by $\otimes$ & opposite - by $\cdot^{opp}$.
This is abelian group.

For Azumaya $\mathcal{A}$, let $[\mathcal{A}]$ be its class in $Br(Z)$.

Fact 0: Let Z_0 be smooth $\mathbb{F}$-variety, $Z = T^*Z_0 \rightsquigarrow$ Azumaya alg'a $\mathcal{D}_Z$ on $Z^{(1)}$. Claim: $p[\mathcal{D}_Z] = 0$ in $Br(Z^{(1)})$ ($p = \operatorname{char} \mathbb{F}$).

Proof: Step 1: Consider Frobenia $Fr: Z \to Z^{(1)}$, $Fr_0: Z_0 \to Z_0^{(1)}$.
Claim: $[Fr^*\mathcal{D}_Z] = 0$ (i.e. $Fr^*\mathcal{D}_Z$ is split).
Consider commutative diagram

$$\begin{tikzcd}
Z \arrow[dr] \arrow[drr, bend left] \arrow[ddr, "Fr"'] & & \\
& Z^{(1)} \times_{Z_0^{(1)}} Z_0 \arrow[r] \arrow[d, "Fr'"] & Z_0 \arrow[d, "Fr_0"] \\
& Z^{(1)} \arrow[r] & Z_0^{(1)}
\end{tikzcd}$$

cotangent projections

Observation: $Fr'^*\mathcal{D}_Z$ is split ([BMR], Prop. 2.2.2.)
$\Rightarrow Fr^*\mathcal{D}_Z$ splits.

Step 2: $Z = Z^{(1)}$, the same scheme $\rightsquigarrow Br(Z) = Br(Z^{(1)})$.
Claim: under this identification Fr^* acts as multiplication by p.
On functions: $Fr^*(f) = f^p$.
Every Azumaya algebra is locally trivial in etale topology;
from Azumaya algebra $\mathcal{A} \rightsquigarrow$ 2-cocycle valued in $\mathbb{G}_m$ in etale topology.
$\rightsquigarrow Br(Z) \overset{(*)}{\hookrightarrow} H^2_{et}(Z, \mathbb{G}_m)$ (see Milne's Etale cohomology)
Multiplication by $a \in \mathbb{Z}$ in $H^2_{et}(Z, \mathbb{G}_m)$ comes from $z \mapsto z^a$ in $\mathbb{G}_m$.
Fr^* acts as taking pth powers on the cocycles, so as mult'n by p in $Br(Z)$. $\square$

(*): from $1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$

Fact 1: Let Z is irreducible & smooth, $Z^\circ \subset Z$ is open ($\neq \emptyset$).
$\rightsquigarrow Br(Z) \hookrightarrow Br(Z^\circ)$.
Proof: see Milne "Etale cohomology," Ch. 4. $\square$

Z is affine $\mathbb{F}$-variety, $\Gamma \curvearrowright Z$, finite group $\rightsquigarrow \pi: Z \to Z/\Gamma$
$\rightsquigarrow \pi^*: Br(Z/\Gamma) \longrightarrow Br(Z)^\Gamma$
For a prime ℓ, let $?[\ell]$ denote the ℓ-torsion part of abelian gr'p $?$.

Fact 2: Suppose $GCD(\ell, |\Gamma|) = 1$. Then $\pi^*: Br(Z/\Gamma)[\ell] \xrightarrow{\sim} Br(Z)[\ell]^\Gamma$
Proof: Lemma 6.5 in [BK]. $\square$

Proof of Prop'n: $V^\circ = \{v \in V \mid Stab_{S_n}(v) = \{1\}\}$ = pairwise distinct pts in $\mathbb{F}^2$,
$\rho: X_{\mathbb{F}}^{(1)} \longrightarrow Y_{\mathbb{F}}^{(1)}$ is iso over $V_{\mathbb{F}}^{\circ(1)}/S_n$. So $\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{S_n}|_{V_{\mathbb{F}}^{\circ(1)}/S_n} \xrightarrow{\sim} \mathcal{D}|_{V_{\mathbb{F}}^{\circ(1)}/S_n}$.

$[\mathcal{D}(\mathfrak{h}_{\mathbb{F}})] \in Br(V_{\mathbb{F}}^{(1)})^{S_n}$, p torsion by Fact 0; $GCD(p, |S_n|) = 1$
Apply Fact 2 to $[\mathcal{D}(\mathfrak{h}_{\mathbb{F}})] \in Br(V^{(1)})[p]^{S_n}$. Take Azumaya algebra $\mathcal{A}$ on $V_{\mathbb{F}}^{(1)}/S_n$ s.t $[\mathcal{A}]$ corresponds to $[\mathcal{D}(\mathfrak{h}_{\mathbb{F}})]$ under isom'm from Fact 2. Note that $[\mathcal{A}|_{V_{\mathbb{F}}^{\circ(1)}/S_n}] = [\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{S_n}|_{V_{\mathbb{F}}^{\circ(1)}/S_n}] = [\mathcal{D}|_{V_{\mathbb{F}}^{\circ(1)}/S_n}]$. So the restr'ns of $\mathcal{D}$ & $\rho^*\mathcal{A}$ to $V_{\mathbb{F}}^{\circ(1)}/S_n \subset X_{\mathbb{F}}^{(1)}$ are Morita equivalent. So $\mathcal{D}$ & $\rho^*\mathcal{A}$ are Morita equivalent $\square$

Rem: Altern. proof of Corollary (Bezrukavnikov - I.L. 13), works for more general Hamilt. reductions

Z_0 smooth $\mathbb{F}$-variety, $\alpha \in \Omega^1(Z_0) \rightsquigarrow Z_0^{(1)} \overset{\iota_\alpha}{\hookrightarrow} Z^{(1)} = T^* Z_0^{(1)}$
$\rightsquigarrow \iota_\alpha^* \mathcal{D}_{Z_0}$, Azumaya alg'a on $Z_0^{(1)}$. There's criterion for such Azumaya algebra to split. We show that $[\mathcal{D}(R_{\mathbb{F}})///_0^\theta G]$ comes from 1-form (contraction of the symplc form on $(\mu^{(1)})^{-1}(0)/G_{\mathbb{F}}^{(1)}$ & the vector field coming from $\mathbb{F}^\times$-action). When restricting to neigh'd of 0, get splitting (Sect. 7.2 in the paper).

3) Morita equivalences for $\mathcal{D}(\mathfrak{h})^{S_n}$ & relatives.

Let $e \in \mathbb{F}S_n$ be trivial idempotent.

$$e(\mathcal{D}(\mathfrak{h}_{\mathbb{F}}) \# S_n)e \xleftarrow{\sim} \mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{S_n}, \text{ isom'm of algebras}$$
$$e\alpha = \alpha e \longleftarrow\!\shortmid \alpha$$

$(\mathcal{D}(\mathfrak{h}_{\mathbb{F}}) \# S_n)e \xleftarrow{\sim} \mathcal{D}(\mathfrak{h}_{\mathbb{F}}) \rightsquigarrow$
$\mathcal{D}(\mathfrak{h}_{\mathbb{F}}) \# S_n$-$\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{S_n}$-bimodule structure on $\mathcal{D}(\mathfrak{h}_{\mathbb{F}})$.

Prop'n 1: This is Morita equivalence bimodule.

Proof: $\mathcal{A}_{\mathbb{F}}^1 = \mathcal{D}(\mathfrak{h}_{\mathbb{F}}) \# S_n$, $\mathcal{A}_{\mathbb{F}}^2 = \mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{S_n}$; $\mathcal{B}_{\mathbb{F}} = \mathcal{D}(\mathfrak{h}_{\mathbb{F}})$

$\mathrm{Hom}_{\mathcal{A}_{\mathbb{F}}^1}(\mathcal{B}_{\mathbb{F}}, \bullet): \mathcal{A}_{\mathbb{F}}^1\text{-Mod} \to \mathcal{A}_{\mathbb{F}}^2\text{-Mod}$ is isomorphic to $e\bullet$

so is Serre quotient functor w. right inverse is $\mathcal{B}_{\mathbb{F}} \otimes_{\mathcal{A}_{\mathbb{F}}^2} \bullet$

To show these are equivalences $\Leftrightarrow \{M_{\mathbb{F}} \mid eM_{\mathbb{F}} = 0\} = 0 \Leftrightarrow$

$\mathcal{A}_{\mathbb{F}}^1 e \mathcal{A}_{\mathbb{F}}^1 = \mathcal{A}_{\mathbb{F}}^1 \Leftarrow \mathcal{A}_{\mathbb{C}}^1 e \mathcal{A}_{\mathbb{C}}^1 = \mathcal{A}_{\mathbb{C}}^1$

exercise

So Prop'n will follow from the next lemma $\square$

Lemma: $\mathcal{A} := \mathcal{D}(\mathfrak{h}_{\mathbb{C}})$. Then $\mathcal{A} \# S_n$ is a simple algebra.

Proof: $\mathcal{A} \# S_n = \bigoplus_{\sigma \in S_n} \mathcal{A}\sigma$, as $\mathcal{A}$-bimodule.

Claims: • $\mathcal{A}\sigma$ is simple $\mathcal{A}$-bimodule ($\Leftarrow \mathcal{A}$ is simple)

• $\mathcal{A}\sigma \not\cong \mathcal{A}\sigma'$ if $\sigma \neq \sigma'$ (follows from the next exercise)

Exercise: Centralizer of $\mathcal{A}$ in $\mathcal{A}\sigma = \begin{cases} \mathbb{C}, & \sigma = 1 \\ \{0\}, & \sigma \neq 1. \end{cases}$

$\overset{\shortparallel}{\{m \in \mathcal{A}\sigma \mid am = ma \ \forall a \in \mathcal{A}\}}$

Therefore every $\mathcal{A}$-sub-bimodule of $\mathcal{A} \# S_n$ is $\bigoplus$ of some $\mathcal{A}\sigma$.

Right S_n-action permutes $\mathcal{A}\sigma$'s transitively $\Rightarrow$ every 2-sided ideal of $\mathcal{A} \# S_n$ is $\mathcal{A} \# S_n$ or $\{0\}$ $\square$

Prop'n 2: $\exists$ $k > 0$ & idempotent $\varepsilon \in Mat_k(\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{\wedge_0 S_n})$ s.t.

$$\varepsilon\, Mat_k(\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{\wedge_0 S_n})\,\varepsilon \xrightarrow{\sim} \mathbb{F}[V^{(1)}]^{\wedge_0} \# S_n,$$

an isom'm of $\mathbb{F}[V^{(1)}/S_n]^{\wedge_0}$.

Proof:

Step 1: $\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{\wedge_0}$ splits as Azumaya algebra over $\mathbb{F}[V^{(1)}]^{\wedge_0}$ (from lifting of idempotents); S_n is reductive gr'p. So splitting can choosen S_n-equivariantly: pick splitting bundle $\mathcal{F}$ of the form $Q \otimes \mathbb{F}[V^{(1)}]^{\wedge_0}$, where Q is an S_n-module & $S_n \curvearrowright Q \otimes \mathbb{F}[V^{(1)}]^{\wedge_0}$ is diagonal $\rightsquigarrow$ S_n-equivariant isom'm

$$\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{\wedge_0} \xrightarrow{\sim} End_{\mathbb{F}[V^{(1)}]^{\wedge_0}}(Q \otimes \mathbb{F}[V^{(1)}]^{\wedge_0}) \qquad (i)$$

Step 2: **Exercise:** $\mathbb{F}[V^{(1)}]^{\wedge_0} \# S_n \longrightarrow End_{\mathbb{F}[V^{(1)}/S_n]^{\wedge_0}}(\mathbb{F}[V^{(1)}]^{\wedge_0}) \qquad (ii)$

is an isomorphism.

Step 3: Combine (i) & (ii):

$$\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{\Lambda_0} \# S_n \xrightarrow{\sim} \mathrm{End}_{\mathbb{F}[V^{(1)}]^{\Lambda_0}}(Q \otimes \mathbb{F}[V^{(1)}]^{\Lambda_0}) \# S_n \xrightarrow{\sim}$$

$$(\mathrm{End}(Q) \otimes \mathbb{F}[V^{(1)}]^{\Lambda_0}) \# S_n \xrightarrow{\sim} [S_n \to \mathrm{End}(Q)]$$

$$\mathrm{End}(Q) \otimes (\mathbb{F}[V^{(1)}]^{\Lambda_0} \# S_n) \xrightarrow{\sim} (ii)$$

$$\mathrm{End}(Q) \otimes \mathrm{End}_{\mathbb{F}[V^{(1)}/S_n]^{\Lambda_0}}(\mathbb{F}[V^{(1)}]^{\Lambda_0}) \xrightarrow{\sim} \mathrm{End}_{\mathbb{F}[V^{(1)}/S_n]^{\Lambda_0}}(Q \otimes \mathbb{F}[V^{(1)}]^{\Lambda_0}).$$

Step 4: Use Prop'n 1: $\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{\Lambda_0 S_n} = e(\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{\Lambda_0} \# S_n)e \xrightarrow{\sim}$

$$e\,\mathrm{End}_{\mathbb{F}[V^{(1)}/S_n]^{\Lambda_0}}(Q \otimes \mathbb{F}[V^{(1)}]^{\Lambda_0})e =$$

$$\mathrm{End}_{\mathbb{F}[V^{(1)}/S_n]^{\Lambda_0}}(e(Q \otimes \mathbb{F}[V^{(1)}]^{\Lambda_0}))$$

What remains to show is that $\mathbb{F}[V^{(1)}]^{\Lambda_0}$ & $e(Q \otimes \mathbb{F}[V^{(1)}]^{\Lambda_0})$ have the same indecomposable summands (but w. different mult's).

Step 5: a) $\forall \tau \in \mathrm{Irr}(S_n)$, $\mathrm{Hom}_{S_n}(\tau, \mathbb{F}[V^{(1)}]^{\Lambda_0})$ is indecomposable summand of $\mathbb{F}[V^{(1)}]^{\Lambda_0}$.

b) # indec. summands in $\mathbb{F}[V^{(1)}]^{\Lambda_0}$ is $\#\mathrm{Irr}(S_n)$:

indec. summands $\xleftrightarrow{\sim}$ indec. projectives in $\mathrm{End}_{\mathbb{F}[V^{(1)}/S_n]^{\Lambda_0}}(\mathbb{F}[V^{(1)}]^{\Lambda_0}) =$

$= \mathbb{F}[V^{(1)}]^{\Lambda_0} \# S_n \xleftrightarrow{\sim}$ simple $\mathbb{F}[V^{(1)}]^{\Lambda_0} \# S_n$-modules = simple $\mathbb{F}S_n$-modules.

c) Same reasoning for $\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{\Lambda_0 S_n}$: indec. summands in

$e(Q \otimes \mathbb{F}[V^{(1)}]^{\Lambda_0}) \xleftrightarrow{\sim}$ simple $\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{\Lambda_0 S_n}$-modules $\xleftrightarrow{\sim}$

Morita equiv

simple $\mathcal{D}(\mathfrak{h}_{\mathbb{F}})^{\Lambda_0} \# S_n$-modules $\xleftrightarrow{\sim}$ simple $\mathbb{F}[V^{(1)}]^{\Lambda_0} \# S_n$-modules

$\overset{\sim}{\longleftrightarrow} \mathrm{Irr}(S_n)$.

d) All indecomposables in $e(Q \otimes \mathbb{F}[V^{(1)}]^{\wedge_0})$ are of the form $\mathrm{Hom}_{S_n}(\tau, \mathbb{F}[V^{(1)}]^{\wedge_0})$. $\square$