RATIONAL REPRESENTATIONS IN POSITIVE CHARACTERISTIC

In this note we discuss several basic results about the rational representations of algebraic groups in characteristic p.

Contents

Throughout this note, p is a prime number, $\mathbb{F}$ is an algebraically closed field of characteristic p, G is a connected reductive algebraic group over $\mathbb{F}$. For example, we can take $G = \mathrm{GL}_n(\mathbb{F}), \mathrm{SL}_n(\mathbb{F}), \mathrm{SO}_n(\mathbb{F}), \mathrm{Sp}_{2n}(\mathbb{F})$. By $G_{\mathbb{C}}$ we denote the corresponding group over $\mathbb{C}$. We write $\mathfrak{g}$ for the Lie algebra of G (still over $\mathbb{F}$).

We consider the category $\mathrm{Rep}(G)$ of finite dimensional rational (=algebraic) representations of G. It sits inside the category $\mathrm{Rep}_\infty(G)$ of all rational representations (i.e., representations which are unions of their finite dimensional rational subrepresentations).

We write Λ for the weight lattice of G (=the character lattice of T) and Λ^+ for the submonoid of dominant weights in Λ. As usual, $T \subset B$ stand for a maximal torus and a Borel subgroup of G and W denotes the Weyl group. We write $U := R_u(B)$ for the unipotent radical of B and U^- for the opposite maximal unipotent subgroup.

We write $\alpha_1, \ldots, \alpha_r$ for the simple roots of G. Furthermore, let α_0 denote the negative root such that $\alpha_0^\vee$ is minimal. So $\alpha_0, \ldots, \alpha_r$ are the simple *affine* roots. We write w_0 for the longest element of W.

1. Families of rational representations

The goal of this section is to produce three families of objects in $\mathrm{Rep}(G)$ indexed by Λ^+, the dual Weyl modules, the Weyl modules and the simple modules.

1.1. **Weights.** It turns out that for the tori the representation theory over $\mathbb{F}$ is not different from that over $\mathbb{C}$.

Lemma 1.1. *Every object of* $\mathrm{Rep}(T)$ *is completely reducible, and the irreducibles are precisely the characters.*

So, for every $V \in \mathrm{Rep}(G)$, we have the weight decomposition $V = \bigoplus_{\chi\in\Lambda} V_\chi$, where

$$V_\chi := \{v \in V | t.v = \chi(t)v, \forall t \in T\}.$$

Note that since $W = N_G(T)/T$, the spaces V_χ and $V_{w\chi}$ are isomorphic for all $\chi \in \Lambda, w \in W$.

1.2. **Induction.** To classify the irreducibles in $\mathrm{Rep}(G)$ we will need the induction functors.

Let H be an algebraic subgroup of an algebraic group G (not necessarily reductive) and $M \in \mathrm{Rep}(H)$ (more generally, we can consider $M \in \mathrm{Rep}_\infty(H)$). Then we can define the induced representation $\mathrm{Ind}_H^G(M)$ as $\Gamma(G/H, G \times^H M)$, where $G \times^H M$ is the homogeneous bundle on G/H with fiber U, in other words, $G \times^H M = (G \times M)/H$. Then we have the Frobenius reciprocity:

$$\mathrm{Hom}_H(V,U) = \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(U)), \forall V \in \mathrm{Rep}(G), U \in \mathrm{Rep}(H). \tag{1.1}$$

In particular, we see that $\mathbb{F}[G] = \mathrm{Ind}_1^G \mathbb{F}$ is an injective generator in $\mathrm{Rep}_\infty(G)$.

Of course, $\mathrm{Ind}_H^G(M)$ is infinite dimensional, in general. It is finite dimensional, provided $\dim M < \infty$ and G/H is projective, equivalently, H is a parabolic subgroup. In what follows, we will mostly consider the situation when $H = B$.

1.3. **(Dual) Weyl modules.** For $\lambda \in \Lambda^+$ by λ^* we denote the dual highest weight, i.e., $\lambda^* = -w_0(\lambda)$.

Definition 1.2. The dual Weyl module $M(\lambda)$ is $\mathrm{Ind}_B^G(\mathbb{F}_{-\lambda^*})$.

Example 1.3. Let $G = \mathrm{SL}_2(\mathbb{F})$. Then $G/B = \mathbb{P}^1, \Lambda^+ = \mathbb{Z}_{\geqslant 0}$ and $G \times^B \mathbb{F}_{-n} = \mathcal{O}(n)$ (here we write $\mathbb{F}_{-n}$ for the 1-dimensional space equipped with a $T = \mathbb{F}^\times$-action via $t \mapsto t^{-n}$). So $M(n) = \Gamma(\mathbb{P}^1, \mathcal{O}(n)) = \mathbb{F}[x,y]_n$, the space of homogeneous degree n polynomials with its natural representation of $\mathrm{SL}_2(\mathbb{F})$.

So

$$\mathrm{Hom}_G(V, M(\lambda)) = \mathrm{Hom}_B(V, \mathbb{F}_{-\lambda^*}). \tag{1.2}$$

Definition 1.4. The Weyl module $W(\lambda)$ is $M(\lambda^*)^*$.

So

$$\mathrm{Hom}_G(W(\lambda), V) = \mathrm{Hom}_B(\mathbb{F}_\lambda, V)(= \{v \in V^U, tv = \lambda(t)v, \forall t \in T\}). \tag{1.3}$$

Note that (1.3) is completely analogous to the universal property of Verma modules (and 1.2 is analogous to the defining property of dual Verma modules).

One can show that $M(\mu)^U$ is one-dimensional and the T-weight is μ (using the Bruhat decomposition). Hence

$$\dim \mathrm{Hom}_G(W(\lambda), M(\mu)) = \delta_{\lambda,\mu}. \tag{1.4}$$

Exercise 1.5. Deduce that $\dim W(\lambda)_\lambda = \dim M(\lambda)_\lambda = 1$.

Remark 1.6. We have $M_{\mathbb{C}}(\lambda) = W_{\mathbb{C}}(\lambda) = L_{\mathbb{C}}(\lambda)$. Over $\mathbb{F}$, even $M(\lambda) \cong W(\lambda)$ may fail. For example, consider the case of $G = \mathrm{SL}_2(\mathbb{F})$ and $\lambda = p$. Here $M(\lambda) = \mathbb{F}[x, y]_p$ is not simple: it has a two-dimensional submodule, namely, $L(p) := \mathrm{Span}_{\mathbb{F}}(x^p, y^p)$. On the other hand, $\mathrm{Hom}_G(\mathbb{F}[x, y]_p, L(p)) = 0$ (an exercise). Since $\mathrm{Hom}_G(W(p), L(p)) = \mathrm{Hom}_G(L(p), \mathbb{F}[x, y]_p)^* \neq \{0\}$, we arrive at $W(p) \ncong \mathbb{F}[x, y]_p$.

1.4. **Classification of simples.** We have the following result.

Theorem 1.7. *For each $\lambda \in \Lambda^+$, there is a unique simple representation $L(\lambda) \in \mathrm{Rep}(G)$ with highest weight λ. Moreover, $L(\lambda)$ is a unique irreducible subrepresentation of $M(\lambda)$ and a unique irreducible quotient of $W(\lambda)$.*

Exercise 1.8. Prove this theorem using the W-invariance of the set of weights and formulas (1.2),(1.3) and (1.4).

2. Steinberg tensor product theorem

The irreducible representations of G have a certain inductive structure arising from the Frobenius epimorphism. This structure is described by the Steinberg tensor product theorem.

2.1. **Frobenius epimorphism.** For an $\mathbb{F}$-scheme X we can define its *Frobenius twist* $X^{(1)}$, the same $\mathbb{Z}$-scheme as X but with $\mathbb{F}$-multiplication twisted by Fr^{-1}. This gives rise to the Frobenius morphism $\mathrm{Fr}_X : X \to X^{(1)}$ whose pullback is given by $f \mapsto f^p$. For example, $(\mathbb{A}^1)^{(1)}$ is naturally isomorphic to $\mathbb{A}^1$ (the same is true for any scheme defined over $\mathbb{F}_p$) and, under this identification, $\mathrm{Fr}_{\mathbb{A}^1}(z) = z^p$.

For an algebraic group H, the variety $H^{(1)}$ is again an algebraic group and the morphism Fr_H is a group homomorphism.

Example 2.1. Let $G = \mathrm{GL}_n(\mathbb{F})$. Then $G^{(1)} = \mathrm{GL}_n(\mathbb{F})$ and $\mathrm{Fr}_G((g_{ij})) = (g_{ij}^p)$. More generally, any connected reductive group G is defined over $\mathbb{F}_p$ hence $G^{(1)} \cong G$ (the same holds for Borels, parabolics, etc.). The morphism Fr_G is restricted from $\mathrm{GL}_n(\mathbb{F})$.

The morphism Fr_H is dominant and hence is a group epimorphism. In particular, if $L^{(1)}$ is a simple representation of $G^{(1)}$, then $\mathrm{Fr}_G^*(L^{(1)})$ is a simple representation of G. Because of the natural identification $G \cong G^{(1)}$, we can view Fr_G^* as an endofunctor of $\mathrm{Rep}(G)$.

Example 2.2. For the representation $L = \mathrm{Span}_{\mathbb{F}}(x, y)$ of $\mathrm{SL}_2(\mathbb{F})$, we get $\mathrm{Fr}^* L = \mathrm{Span}_{\mathbb{F}}(x^p, y^p)$.

Exercise 2.3. If $L^{(1)}$ has highest weight λ, then $\mathrm{Fr}_G^*(L^{(1)})$ has highest weight $p\lambda$. Shortly, $\mathrm{Fr}_G^*(L(\lambda)) = L(p\lambda)$.

Remark 2.4. We have a functor $\mathrm{Rep}(G) \to \mathfrak{g}$-mod given by differentiation. But this functor no longer distinguishes representations: for any M, it sends $\mathrm{Fr}_G^*(M)$ to a trivial representation of $\mathfrak{g}$.

2.2. **The case of** SL_2**.** Before stating the Steinberg theorem in general, let us consider the case of SL_2.

Lemma 2.5. *Let $i = 0, \dots, p-1$ and M be an irreducible G-module. Then the G-representation $L(i) \otimes \mathrm{Fr}_G^*(M)$ is irreducible.*

Proof. Note that the irreducible representations $L(0), \dots, L(p-1)$ are also irreducible as $\mathfrak{g}$-modules. So every irreducible $\mathfrak{g}$-submodule of $L(i) \otimes \mathrm{Fr}_G^*(M)$ takes the form $L(i) \otimes M_0$ for some $M_0 \subset \mathrm{Fr}_G^*(M)$. If $L(i) \otimes M_0$ is a G-submodule, then $M_0 \subset \mathrm{Fr}_G^*(M)(= M)$ is a $G^{(1)}$-submodule. $\square$

An corollary of this lemma and Theorem 1.7 is the following statement.

Corollary 2.6. *For $i = 0, 1, \dots, p-1, j \in \mathbb{Z}_{\geqslant 0}$ we have $L(i + pj) \cong L(i) \otimes \mathrm{Fr}_G^*(L(j))$.*

Proof. The right hand side is irreducible by Lemma 2.5 and has highest weight $pj + i$. The isomorphism follows from Theorem 1.7. $\square$

Applying this corollary while we can, we get the Steinberg tensor product theorem for SL_2. Let $\lambda \in \Lambda^+ = \mathbb{Z}_{\geqslant 0}$ and consider the p-adic expansion $\lambda = \lambda_0 + p\lambda_1 + \ldots + p^m\lambda_m$ (with $\lambda_0, \dots, \lambda_m \in \{0, \dots, p-1\}$).

Corollary 2.7. *We have*

$$L(\lambda) = L(\lambda_0) \otimes \mathrm{Fr}_G^* L(\lambda_1) \otimes (\mathrm{Fr}_G^*)^2 L(\lambda_2) \otimes \ldots \otimes (\mathrm{Fr}_G^*)^m L(\lambda_m).$$

This result allows to compute the dimension and the character of $L(\lambda)$.

2.3. **Theorem in general.** Assume, to simplify the statements, that G is semisimple and simply connected. This is a reasonable restriction: any connected reductive G_1 admits a surjective central isogeny with finite kernel from $T \times G'$, where $T := Z(G)^\circ$ and G' is semisimple and simply connected.

Set

$$\Lambda_1^+ := \{\lambda \in \Lambda^+ | \langle \lambda, \alpha_i^\vee \rangle < p, \forall i = 1, \dots, r\}.$$

The elements of Λ_1^+ are called the *p-restricted weights.*

Then, every $\lambda \in \Lambda^+$ can be uniquely written as

$$\lambda = \sum_{i=0}^{m} p^i \lambda_i, \quad \lambda_0, \dots, \lambda_m \in \Lambda_1^+. \tag{2.1}$$

Theorem 2.8. *We have $L(\lambda) = L(\lambda_0) \otimes \mathrm{Fr}_G^* L(\lambda_1) \otimes \ldots \otimes (\mathrm{Fr}_G^*)^m L(\lambda_m)$.*

This theorem will be proved later.

Remark 2.9. Unlike in the case of SL_2, the character of $L(\lambda_0)$ is not always the same as in characteristic 0. For example, consider $p = 3, G = \mathrm{SL}_3, \lambda = 3\pi_1$, where π_1 is the fundamental weight. Similarly to the SL_2-case, $M(\lambda) = \mathbb{F}[x_1, x_2, x_3]_3$, the dimension is 10. It has a sub, it is $\mathrm{Span}(x_1^3, x_2^3, x_3^3)$ and has dimension 3. The quotient is seven dimensional, has highest weight $\pi_1 + \pi_2 \in \Lambda_1^+$ (the highest weight vector is $x_1^2 x_2$). The dimension of the irreducible representation with this highest weight in characteristic 0 is 8, it is the adjoint representation.

2.4. **Frobenius kernels.** Let H be a connected algebraic group. Set theoretically, the kernel of $\mathrm{Fr}_H : H \twoheadrightarrow H^{(1)}$ is $\{1\}$ but scheme theoretically it has length $p^{\dim H}$. This normal group subscheme is denoted by H_1 and is called the 1st *Frobenius kernel* (one can also consider *higher Frobenius kernels* H_r of the epimorphisms $\mathrm{Fr}_H^r : H \twoheadrightarrow H^{(r)}$).

The algebra

$$\mathbb{F}[H_1] = \mathbb{F}[H]/(\mathrm{Fr}_H^*(f) | f \in \mathbb{F}[H^{(1)}], f(1_H) = 0)$$

is a finite dimensional Hopf algebra – as a Hopf quotient of the Hopf algebra $\mathbb{F}[H]$ – so it makes sense to define the representations of H_1 as $\mathbb{F}[H_1]$-comodules, equivalently, as modules over the dual Hopf algebra $\mathrm{D}(H_1) := \mathbb{F}[H_1]^*$, a.k.a. the *distribuition algebra.*

Example 2.10. Consider the additive group $H = \mathbb{G}_a$. Then $\mathbb{F}[H] = \mathbb{F}[x], \mathbb{F}[H_1] = \mathbb{F}[x]/(x^p)$. The coproduct Δ on $\mathbb{F}[H]$ is given by the general formula $\Delta(f)(z_1, z_2) := f(z_1, z_2)$ (the equality of functions on $H \times H$), equivalently, on the generator x, $\Delta(x) = x \otimes 1 + 1 \otimes x$. So the coproduct on $\mathbb{F}[H_1]$ is also given by $\Delta(x) = x \otimes 1 + 1 \otimes x \in \mathbb{F}[H_1] \otimes \mathbb{F}[H_1]$. This Hopf algebra is self-dual.

Exercise 2.11. Now let $H = \mathbb{G}_m$. Then $\mathbb{F}[H_1] = \mathbb{F}[x]/(x^p - 1)$ and the coproduct is given by $\Delta(x) = x \otimes x$. Let $h : \mathbb{F}[H_1] \to \mathbb{F}$ be the function defined by $h(x^i) = i$ for $i = 0, \dots, p-1$ so that $h \in D(H_1)$. Then $\mathrm{D}(H_1) = \mathbb{F}[h]/(h^p - h)$ and the coproduct given by $h \mapsto h \otimes 1 + 1 \otimes h$.

Remark 2.12. In both cases, $\mathrm{D}(H_1)$ turns out to be a Hopf quotient of $U(\mathfrak{h})$. This is also the case in general. First of all, there is a natural homomorphism $U(\mathfrak{h}) \to D(H_1)$ that sends $a \in U_{\mathfrak{h}}$ to the functional $\mathbb{F}[H_1] \to \mathbb{F}$ sending f to $(a.f)(1)$. Next, the Lie algebra $\mathfrak{h}$ of H comes with an additional structure, the restricted pth power map $x \mapsto x^{[p]} : \mathfrak{h}^{(1)} \to \mathfrak{h}$ that remembers the structure of an *algebraic* group on H. For example, for $G = \mathrm{GL}_n(\mathbb{F})$, the element $x^{[p]}$ is the pth power of the matrix x. For general H, we can embed $H \hookrightarrow \mathrm{GL}_n(\mathbb{F})$ and then, for $x \in \mathfrak{h}^{(1)}$, the element $x^{[p]}$ is still the pth power of the matrix x. One can show that $x \mapsto x^p - x^{[p]} : \mathfrak{h}^{(1)} \to U(\mathfrak{h})$ is a linear map with central image. The ideal generated by these elements is a Hopf ideal and we can consider the quotient Hopf algebra $U^0(\mathfrak{h}) = U(\mathfrak{h})/(x^p - x^{[p]})$. One can then show that the homomorphism $U(\mathfrak{h}) \to \mathrm{D}(H_1)$ factors through $U^0(\mathfrak{h}) \xrightarrow{\sim} D(H_1)$. This equality describes a connection between the representations of H and $\mathfrak{h}$ in positive characteristic: passing from representations of G to representations of $\mathfrak{g}$ amounts to restricting to a normal subgroup (scheme).

Now let us describe the structure of G_1. Note that the multiplication map defines an open inclusion of schemes (as the open Bruhat cell)

$$U^- \times T \times U \hookrightarrow G$$

This gives rise to an *isomorphism* of schemes (<u>not</u> of group schemes)

$$U_1^- \times T_1 \times U_1 \xrightarrow{\sim} G_1. \tag{2.2}$$

Similarly, if U_α denotes the root subgroup corresponding to α, then

$$\prod_{\alpha>0} U_{\alpha,1} \xrightarrow{\sim} U_1, \prod_{\alpha<0} U_{\alpha,1} \xrightarrow{\sim} U_1^-.$$

2.5. **Representation theory of G_1.** The representation of T_1 are completely reducible and the irreducibles are parameterized by $\Lambda/p\Lambda$. We can still consider the coinduced module

$$M_1(\lambda) := \mathrm{Ind}_{B_1}^{G_1} \mathbb{F}_{-\lambda^*} \tag{2.3}$$

and

$$W_1(\lambda) := M_1(\lambda^*)^*. \tag{2.4}$$

Note that thanks to (2.2), we have $\dim M_1(\lambda) = \dim W_1(\lambda) = p^{\dim U}$.

Similarly to Theorem 1.7, we get the following result.

Proposition 2.13. *For each $\lambda \in \Lambda/p\Lambda$, there is a unique simple representation $L_1(\lambda) \in \mathrm{Rep}(G_1)$ that is a unique irreducible subrepresentation of $M_1(\lambda)$ and a unique irreducible quotient of $W_1(\lambda)$.*

Remark 2.14. $W_1(\lambda)$ and $M_1(\lambda)$ are *baby* Verma and dual Verma modules for $U^0(\mathfrak{g})$. The analogy with the (dual) Vermas suggest to consider the case of $\lambda = (p-1)\rho$ (the most singular block). In the case of $U(\mathfrak{g}_{\mathbb{C}})$, we have $\Delta(-\rho) = \nabla(-\rho) = L(-\rho)$. And in our situation, we have $M_1((p-1)\rho) = W_1((p-1)\rho) = L_1((p-1)\rho)$. This is the so called *Steinberg representation* of G_1. We note that $(p-1)\rho$ is a maximal p-restricted weight.

Remark 2.15. Results of this section generalize to the higher Frobenius kernels. In particular, for $\lambda \in \Lambda/p^r\Lambda$, we have modules $M_r(\lambda), W_r(\lambda)$ and the simples are now classified by $\Lambda/p^r\Lambda$. We also have the irreducible Steinberg representation $L_r((p^r-1)\lambda) = M_r((p^r-1)\lambda)$.

2.6. **Sketch of proof of Theorem 2.8.** We now proceed to proving the Steinberg tensor product theorem. We need to prove the following statement. Then we can proceed as in the case of SL_2.

Proposition 2.16. *Let $\lambda \in \Lambda_1^+$. Then the restriction of $L(\lambda)$ to G_1 is isomorphic to $L_1(\lambda)$.*

Sketch of proof. The proof is in two steps.

Step 1. First, we prove that $L_1(\lambda)$ lifts to a representation of G, the lift is automatically irreducible. The group G acts on G_1 by conjugations and hence permutes the (isomorphism classes of) irreducibles for G_1. Since G is connected, it fixes each irreducible. Hence $L_1(\lambda)$ becomes a projective representation of G. But since G is simply connected, $L(\lambda)$ must be a genuine representation.

Step 2. We need to verify that the highest weight λ' of the resulting irreducible G-representation is λ. Restricting $L(\lambda')$ back to G_1 we see that, $\lambda' - \lambda \in p\Lambda$. So we only need to show that $\lambda' \in \Lambda_1^+$. Assume the contrary, then we can find a simple root α with $\langle \lambda', \alpha^\vee \rangle \geqslant p$. Let us take the highest vector v of $L(\lambda')$. The submodule $D(U_1^-)v = U(\mathfrak{g})v$ coincides with $L(\lambda')$ so must have nonzero weight space of weight $s_\alpha \lambda' = \lambda' - \langle \lambda, \alpha^\vee \rangle \alpha$. The whole situation is Λ-graded and hence this weight space must be already in $D(U_{-\alpha,1})v$. By Example 2.10, the weights in that space are $\lambda' - i\alpha$ for $i = 0, \ldots, p-1$, while $\langle \lambda', \alpha^\vee \rangle \geqslant p$. This contradiction finishes the proof. $\square$

3. Kempf's vanishing

A natural question is to compute the characters of $M(\lambda), W(\lambda)$. Here's the result that we already know for SL_2.

Theorem 3.1. *Let G be a connected reductive group. Then the characters of $M(\lambda), W(\lambda)$ are the same as in characteristic 0, i.e. are given by the Weyl character formula.*

The proof can be easily reduced to the case when G is semisimple and simply connected.

3.1. **Kempf's theorem.** Here is how Theorem 3.1 is proved. It is enough to handle the case of $M(\lambda) = H^0(G/B, \mathcal{O}(\lambda))$. Second, as in characteristic zero, the Euler characteristic $\chi(G/B, \mathcal{O}(\lambda))$ is given by the Weyl character formula. So to prove Theorem 3.1 we need to have the following theorem due to Kempf (also something we already know for SL_2).

Theorem 3.2. *For $\lambda \in \Lambda^+$, we have $H^i(G/B, \mathcal{O}(\lambda)) = 0$ for all $i > 0$.*

We note that since $\mathrm{Rep}_\infty(B)$ has enough injectives, we can define the derived induction functor $R\operatorname{Ind}_B^G : D^+(\mathrm{Rep}_\infty(B)) \to D^+(\mathrm{Rep}_\infty(G))$. It is not hard to show that it restricts to $D^b(\mathrm{Rep}(B)) \to D^b(\mathrm{Rep}(G))$. Observe that

$$R\operatorname{Ind}_B^G(M) = R\Gamma(G/B, G \times^B M), \forall M \in \mathrm{Rep}(B). \tag{3.1}$$

The main ingredient in the proof is the following result.

Proposition 3.3. *Let $r \geqslant 1$. Then, for $M \in \mathrm{Rep}(B)$, we have the following functorial isomorphism:*

$$R\operatorname{Ind}_B^G\left(\mathbb{F}_{-(p^r-1)\rho} \otimes (\mathrm{Fr}_B^*)^r M\right) \cong L((p^r-1)\rho) \otimes (\mathrm{Fr}_G^* R)^r R\operatorname{Ind}_B^G(M). \tag{3.2}$$

Proof. We use the transitivity of induction for the intermediate subgroup (scheme) $H = G_r B$ and also a basic fact that for a group inclusion $H_1 \subset H_2$, we have

$$\operatorname{Ind}_{H_1}^{H_2}(M_2 \otimes M_1) = M_2 \otimes \operatorname{Ind}_{H_1}^{H_2}(M_1), \forall M_1 \in \mathrm{Rep}(H_1), M_2 \in \mathrm{Rep}(H_2). \tag{3.3}$$

An analog of (2.2) for general r yields $H/B = U_r^-$. This is an affine group scheme, so $R\operatorname{Ind}_B^H = \operatorname{Ind}_B^H$. Therefore

$$R\operatorname{Ind}_B^G = R\operatorname{Ind}_H^G \circ \operatorname{Ind}_B^H. \tag{3.4}$$

Apply this equality with the module $\mathbb{F}_{-(p^r-1)\rho} \otimes (\mathrm{Fr}_B^*)^r M \in \mathrm{Rep}(B)$. Note that, since $G_r \lhd H$, for $N \in \mathrm{Rep}(B)$, the G_r-action on $\operatorname{Ind}_{B_r}^{G_r}(N)$ naturally lifts to H and the resulting H-module is naturally identified with $\operatorname{Ind}_B^H(N)$. Also B_r acts trivially on $(\mathrm{Fr}_B^*)^r M$. So $(\mathrm{Fr}_B^*)^r M$ can be viewed as an H-module with trivial G_r-action. Hence

$$\operatorname{Ind}_B^H\left(\mathbb{F}_{-(p^r-1)\rho} \otimes (\mathrm{Fr}_B^*)^r M\right) = \operatorname{Ind}_B^H\left(\mathbb{F}_{-(p^r-1)\rho}\right) \otimes (\mathrm{Fr}_B^*)^r M = L((p^r-1)\rho) \otimes (\mathrm{Fr}_B^*)^r M.$$

And the following equality finishes the proof:

$$\begin{aligned} &R\operatorname{Ind}_H^G(L((p^r-1)\rho) \otimes (\mathrm{Fr}_B^*)^r M) = L((p^r-1)\rho) \otimes R\operatorname{Ind}_{B^{(r)}}^{G^{(r)}}(M) = \\ &L((p^r-1)\rho) \otimes (\mathrm{Fr}_G^*)^r R\operatorname{Ind}_B^G(M). \end{aligned}$$

□

Sketch of proof of Theorem 3.2. Note that $\mathcal{O}(\lambda)$ is ample for $\lambda \in \Lambda^+$ and apply Proposition 3.3 to $M = \mathbb{F}_{-\lambda^*}$. The left hand side has no higher cohomology as long as r is large enough so neither does the right hand side. It follows that $R\operatorname{Ind}_B^G(M)$ has no higher cohomology, and we are done. □

Exercise 3.4. Show that if $\langle \lambda, \alpha^\vee \rangle = -1$ for some simple root α, then $R\operatorname{Ind}_B^G(\mathbb{F}_{-\lambda}) = 0$. Hint: use the transitivity of induction for the minimal parabolic P_α associated with α.

In particular, we see that $R\operatorname{Ind}_B^G(\mathbb{F}_{-\lambda})$ has no higher cohomology as long as $\langle \lambda + \rho, \alpha_i^\vee \rangle \geqslant 0, \forall i = 1, \ldots, r$.

3.2. **Ext vanishing and highest weight structure.** Recall (1.4). It turns out that the higher Ext's vanish.

Proposition 3.5. *We have* $\mathrm{Ext}_G^i(W(\lambda), M(\mu)) = 0$ *for all* $i > 0, \lambda, \mu \in \Lambda^+$.

Sketch of proof. Note that $\bullet^*$ is a contravariant self-equivalence of $\mathrm{Rep}(G)$. It follows that $\mathrm{Ext}^i(W(\lambda), M(\mu)) = \mathrm{Ext}^i(W(\mu^*), M(\lambda^*))$. Therefore in the proof it is enough to assume that $\lambda \not\geqslant \mu$. Since $M(\mu) = R\operatorname{Ind}_B^G \mathbb{F}_{-\mu}$ we see that

$$\mathrm{Ext}_G^i(W(\lambda), M(\mu)) = \mathrm{Ext}_B^i(W(\lambda), \mathbb{F}_{-\mu^*}).$$

All T-weights in $W(\lambda)$ are $\geqslant -\lambda^*$. On the other hand, all T-weights in $\mathbb{F}[B]$, an injective hull in $\mathrm{Rep}_\infty(B)$, are nonpositive linear combinations of positive roots. So all weights in an injective resolution are $\leqslant -\mu^*$ with μ^* only in degree 0. It follows that $\mathrm{Ext}^i_B(W(\lambda), \mathbb{F}_{-\mu^*}) = 0$ for $i > 0$. $\square$

A consequence of Proposition 3.5 is that $\mathrm{Rep}(G)$ is kind of a highest weight category (it is not a highest weight category in the literal sense, since it has infinitely many simples). Namely, fix $\mu \in \Lambda^+$ and consider the full subcategory $\mathrm{Rep}_{\leqslant\mu}(G) \subset \mathrm{Rep}(G)$ consisting of all V with all weights $\leqslant \mu$.

Exercise 3.6. $\mathrm{Rep}_{\leqslant\mu}(G)$ is a highest weight category with simples $L(\lambda)$, standards $W(\lambda)$ and costandards $M(\lambda)$ for $\lambda \leqslant \mu$ (and the standard order).

4. Linkage principle

Our question for this section: which simples occur with nonzero multiplicity in $M(\lambda)$ (equivalently, $W(\lambda)$ – these objects have the same characters hence the same K_0-classes)? Let us recall how the answer to a similar question looks like for the category $\mathcal{O}$. Namely, we have the Bruhat order $\preceq$ on Λ: $\mu \preceq \lambda$ if there is a sequence $\lambda_0 = \lambda, \lambda_1, \ldots, \lambda_k = \mu$ such that $\lambda_{i+1} < \lambda_i$ (in the usual order) and $\lambda_{i+1} = s_{\beta_i} \cdot \lambda_i$ for some (not necessarily simple) root β_i. Then $[\nabla(\lambda) : L(\mu)] \neq 0 \Rightarrow \mu \preceq \lambda$.

It turns out that the answer for $\mathrm{Rep}(G)$ looks similar: but we need to consider the affine Weyl group instead of a finite one.

4.1. **Main result.**

Consider the (dual) affine Weyl group $W^a := W \ltimes \Lambda_r$, where we write Λ_r for the root lattice, and its *p-rescaled* dot-action on Λ given as follows:

$$w \cdot_p \lambda \text{ is as before }, t_\nu \cdot_p \lambda = \lambda + p\mu, w \in W, \nu \in \Lambda_r.$$

Note that this action still has fundamental alcove, it is given by

$$A^+ := \langle \lambda \in \Lambda | \langle \lambda + \rho, \alpha_i^\vee \rangle \geqslant 0, \forall i = 1, \ldots, r, \langle \lambda + \rho, \alpha_0^\vee \rangle \geqslant -p\}.$$

We define the *linkage order* $\uparrow$ on Λ as follows: $\lambda \uparrow \mu$ if there are $\lambda_0 = \mu, \lambda_1, \ldots, \lambda_k = \lambda \in \Lambda$ and (not necessarily simple) <u>affine</u> reflections $s_0, \ldots, s_{k-1}$ such that $\lambda_i = s_{i-1} \cdot_p \lambda_{i-1}$ and $\lambda_i \leqslant \lambda_{i-1}$ for all $i = 1, \ldots, k$. Note that $\lambda \uparrow \mu$ implies $\lambda \leqslant \mu$ and $\mu \in W^a \cdot_p \lambda$ (but not vice versa).

Exercise 4.1. We have the following:

- $\uparrow$ induces an order on the set of alcoves and A^+ is the minimal alcove inside $\Lambda^+ - \rho$.
- If $\lambda \in \Lambda$ and α is a positive root, then $\lambda - p\alpha \uparrow \lambda$.
- For $\nu \in \Lambda$ we have $\lambda \uparrow \mu \Leftrightarrow (\lambda + p\nu) \uparrow (\mu + p\nu)$.
- If $\langle \lambda + \rho, \alpha_i^\vee \rangle > 0$ for all $i = 1, \ldots, r$, then $w \cdot \lambda \uparrow \lambda$ for all $w \in W$.

Theorem 4.2. *If the multiplicity $[M(\mu) : L(\lambda)]$ of $L(\lambda)$ in $M(\mu)$ is nonzero, then $\lambda \uparrow \mu$.*

Note that, by weight considerations, we already know that $[M(\mu) : L(\lambda)] \neq 0 \Rightarrow \lambda \leqslant \mu$ and $[M(\mu) : L(\mu)] = 1$. We are not going to prove Theorem 4.2.

4.2. **Block decomposition.**

Corollary 4.3. *If* $\operatorname{Ext}^1(L(\lambda), L(\mu)) \neq 0$, *then* $\mu \in W^a \cdot_p \lambda$.

Proof. Let $0 \to L(\mu) \to V \to L(\lambda) \to 0$ be a nonsplit exact sequence. We need to consider two cases: $\lambda < \mu$ and $\lambda \not< \mu$. If $\lambda < \mu$, then $\operatorname{Hom}_G(L(\mu), M(\lambda)) = 0$, hence $\mathbb{F} = \operatorname{Hom}_G(L(\lambda), M(\lambda)) \xrightarrow{\sim} \operatorname{Hom}_G(V, M(\lambda))$. Since the exact sequence doesn't split, we see that a nonzero homomorphism $V \to M(\lambda)$ must be injective. Now apply Theorem 4.2.

Now consider the case $\lambda \not< \mu$. In this case, $V_\lambda^U \xrightarrow{\sim} L(\lambda)_\lambda^U = \mathbb{F}$ so we have a nonzero homomorphism $W(\lambda) \to V$. Similarly to the previous paragraph, we see that

$$W(\lambda) \twoheadrightarrow V \Rightarrow [W(\lambda) : L(\mu)] \neq 0 \Rightarrow \mu \in W^a \cdot_p \lambda.$$

□

Exercise 4.4. Show that $\operatorname{Ext}^1_G(L(\lambda), L(\lambda)) = 0$ for all $\lambda \in \Lambda$.

In fact, there is an alternative proof of the corollary under some restrictions on p that we would like to sketch as it doesn't use Theorem 4.2.

Sketch of alternative proof. Every $M \in \operatorname{Rep}(G)$ is also a $U(\mathfrak{g})$-module. The center $U(\mathfrak{g})^G$ is identified with $S(\mathfrak{h})^W$ for the dot action of W on $\mathfrak{h}$ (this holds in any characteristic). And the invariants still separate orbits. It follows that if $\mu \notin W^a \cdot_p \lambda$, then there is an element $z \in U(\mathfrak{g})^G$ that acts on $L(\lambda), L(\mu)$ by different scalars. So if $\operatorname{Ext}^1(L(\lambda), L(\mu)) \neq 0$, then λ, μ lie in the same orbit of the *extended* affine Weyl group, $W \ltimes \Lambda$. Also $\lambda - \mu \in \Lambda_r$. So as long p does not divide $|\Lambda/\Lambda_r|$, the corollary follows. □

Now pick $\zeta \in A^+$ and let $\operatorname{Rep}_\zeta(G)$ be the Serre span of $L(\lambda)$ (equivalently, $M(\lambda)$ or $W(\lambda)$) with $\lambda \in W^a \cdot_p \zeta$. Corollary 4.3, we have the direct sum decomposition

$$\operatorname{Rep}(G) = \bigoplus_{\zeta \in A^+} \operatorname{Rep}_\zeta(G). \tag{4.1}$$

In particular, we have the *principal block* $\operatorname{Rep}_0(G)$. We would like to point out, however, that we do not claim that the categories $\operatorname{Rep}_\kappa(G)$ are indecomposable. In particular, $\operatorname{Rep}_{-\rho}(G)$ is decomposable (it is actually equivalent to the entire category $\operatorname{Rep}(G)$).

Remark 4.5. In fact, if $\lambda \in W^a \cdot_p \zeta$, then $R^i \operatorname{Ind}_B^G(\mathbb{F}_{-\lambda}) \in \operatorname{Rep}_\zeta(G)$. To see this one either generalizes Theorem 4.2 to the higher cohomology or (for $p > h := 1 - \langle \rho, \alpha_0^\vee \rangle$ – the Coxeter number) runs an argument similar to the alternative proof (using sheaves of twisted differential operators and their connections to Harish-Chandra central reductions of $U(\mathfrak{g})$).

Let us describe the possible highest weights in each $\operatorname{Rep}_\kappa(G)$. Let us start with the case when ζ is *regular*, meaning that it lies in the interior of A^+ (such ζ exists if and only if $p \geqslant h$). Then, for $u \in W^a$, the inclusion $u \cdot_p \zeta \in \Lambda^+$ is equivalent to u being shortest in Wu.

If ζ is *singular* (=not regular), the stabilizer $\operatorname{Stab}_{W^a}(\zeta)$ is a finite parabolic subgroup in W^a. The inclusion $u \cdot_p \zeta \in \Lambda^+$ is equivalent to u being shortest in Wu and the stabilizer of Wu in $\operatorname{Stab}_{W^a}(\zeta)$ is trivial. In this case we can uniquely choose u so that it is still shortest in Wu but longest in $u \operatorname{Stab}_{W^a}(\zeta)$.

Example 4.6. Let us consider the case of SL_2. Here $A^+ = \{n \in \mathbb{Z} | -1 \leqslant n \leqslant p-1\}$. We have two singular values: -1 and $p-1$. For $\zeta = -1$ our set of u consists of $\Lambda_r^+ \subset W^a$. Equivalently, these are the elements whose reduced expression starts with s_0 and ends with s_1. For $\zeta = p-1$ we need all u that start and end with s_0.

5. Translation and reflection functors

Translation and reflection functors are powerful tools to study the BGG category $\mathcal{O}$. Thanks to (4.1), we have complete analogs of these functors for $\operatorname{Rep}(G)$.

5.1. **Translation functors.** Let pr_ζ be the projection $\operatorname{Rep}(G) \twoheadrightarrow \operatorname{Rep}_\zeta(G)$.

Definition 5.1. Let $\zeta, \eta \in A^+$. Let ν be the dominant weight in $W(\eta - \zeta)$. Define the *translation functor* $T_{\zeta\to\eta} : \operatorname{Rep}_\zeta(G) \to \operatorname{Rep}_\eta(G)$ by

$$T_{\zeta\to\eta}(V) = \operatorname{pr}_\eta(L(\nu) \otimes V).$$

Let us list some properties of the translation functors.

1) $T_{\zeta\to\eta}$ is biadjoint to $T_{\eta\to\zeta}$ and is exact. This is proved in the same as for the category $\mathcal{O}$.

2) Let τ be the Cartan involution of G. For $V \in \operatorname{Rep}(G)$, set $V^\vee = {}^\tau V^*$, where the right superscript means the twist by τ. For this twisted duality we have $L(\lambda)^\vee = L(\lambda)$. In particular, $\bullet^\vee$ fixes all the blocks $\operatorname{Rep}_\zeta(G)$.

Exercise 5.2. We have a natural isomorphism $T_{\zeta\to\eta}(\bullet^\vee) = T_{\zeta\to\eta}(\bullet)^\vee$.

3) Let us investigate the behavior of $T_{\zeta\to\eta}$ on the dual Weyl modules. Suppose that η lies in the closure of the face of A^+ containing ζ. The following property is proved in the same way as for the Verma modules (or, even better, the parabolic Verma modules) in the category $\mathcal{O}$.

Proposition 5.3. *Let $\lambda = u \cdot_p \zeta$ for $u \in W^a$. Set $\mu := u \cdot_p \eta$. Then $T_{\zeta\to\eta}M(\lambda) = M(\mu)$ (where we set $M(\mu) = 0$ if μ is not dominant).*

Note that, thanks to 2), the similar claim holds for Weyl modules.

4) When $F_\zeta = F_\eta$, then $T_{\zeta\to\eta}$ and $T_{\eta\to\zeta}$ are mutually inverse equivalences. This follows from 1) and 3).

5) Now suppose that η lies in the closure of the face of A^+ containing ζ. Then $T_{\eta\to\zeta}$ maps (dual) Weyl modules to modules that admit (dual) Weyl filtration. This follows from 1),3) and the equality $\dim \operatorname{Ext}^i(W(\lambda), M(\mu)) = \delta_{i0}\delta_{\lambda\mu}$.

5.2. **Reflection functors and tilting modules.** Assume $p \geqslant h$ so that 0 is regular. Consider the special case of 5): $\zeta = 0$ and $\eta_i \in A^+, i = 0, \ldots, r$ is such that $\operatorname{Stab}_{W^a}(\eta_i) = \langle s_i \rangle$.

Definition 5.4. We define the reflection functor Θ_i as $T_{\eta_i\to 0} \circ T_{0\to\eta_i}$.

This is a self-biadjoint functor that intertwines the twisted duality functor $\bullet^\vee$. For $u \in W^a$ shortest in Wu the property 5) gives the following:

- $\Theta_i M(u \cdot_p 0) = 0$ if us_i is no longer shortest in Wus_i,
- if us_i is shortest in Wus_i, then $\Theta_i M(u \cdot_p 0)$ is filtered with $M(u \cdot_p 0)$ and $M(us_i \cdot_p 0)$.

Let us state the latter result on the level of K_0. Consider the right W^a-module $\operatorname{sgn} \otimes_{\mathbb{Z}W} \mathbb{Z}W^a$ (a.k.a. the anti-spherical module). We identify $K_0(\operatorname{Rep}_0(G))$ with $\operatorname{sgn} \otimes_{\mathbb{Z}W} \mathbb{Z}W^a$ by sending $[\Delta(u \cdot_p 0)]$ to the image of u. Then the class of Θ_i in K_0 satisfies $[\Theta_i] = (1 + s_i)$, the equality of operators on K_0. One can say that $\operatorname{Rep}_0(G)$ with the reflection functors categorifies the anti-spherical module.

Finally, let us discuss an application of reflection functors to tilting modules (i.e. G-representations that are both Weyl filtered and dual Weyl filtered). For general highest weight reasons, we have one indecomposable tilting $T(\lambda)$ per highest weight λ of $\operatorname{Rep}_0(G)$.

Note that 0 is the minimal highest weight in $\mathrm{Rep}_0(G)$. So $W(0) = M(0) = L(0)$ (or we can simply recall that $M(0)$ is one-dimensional). This is a tilting object. Every Θ_i maps a (dual) Weyl filtered module to a (dual) Weyl filtered module, hence it maps a tilting to a tilting. The following proposition mirrors the corresponding property of projectives in the BGG category $\mathcal{O}$.

Proposition 5.5. *Let $u \in W^a$ be minimal in Wu and $u = s_{i_1} \dots s_{i_k}$ be a reduced expression. Set $\lambda = u \cdot_p 0$. Then $T(\lambda)$ appears as a summand of $\Theta_{i_1} \dots \Theta_{i_k} T(0)$ with multiplicity 1, while all other summands $T(\mu)$ satisfy $\mu \uparrow \lambda$.*