Problem set 1

Due date: Feb 5

January 22, 2018

1. If $\{h_n\}$ is a sequence in a Hilbert space $\mathcal{H}$ such that $\sum_n \|h_n\| < \infty$, then show that h_n converges.

2. Suppose that E is a linear subspace of a Hilbert space $\mathcal{H}$, then show that the closure of E is also a linear subspace

3. Suppose that E is a subspace of a Hilbert space $\mathcal{H}$, then show that $(E^\perp)^\perp$ is the closure of the span of elements in E, i.e.

$$(E^\perp)^\perp = \overline{\left\{ \sum_{j=1}^{N} c_j f_j \,, \quad f_j \in E \right\}}$$

4. Suppose that $\mathcal{H} = \ell^2(\mathbb{N})$.

   (a) Show that if $\{a_n\} \in \mathcal{H}$, then the power series $\sum_{n=1}^\infty a_n z^n$ has radius of convergence at least 1

   (b) For $\lambda < 1$, show that $L(\{a_n\}) := \sum_{n=1}^\infty a_n \lambda^n$ is a bounded linear functional

   (c) Find the element $h_0 \in \mathcal{H}$ such that $L(h) = (h, h_0)$ and find $\|L\|$

5. Let $\mathcal{H}_1 = \mathbb{L}^2([-\pi, \pi])$ be the Hilbert space of functions $F(e^{i\theta})$ on the unit circle with the inner product
$$(F, G) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta}) \overline{G(e^{i\theta})} d\theta \,.$$
Let $\mathcal{H}_2$ be the space $\mathbb{L}^2(\mathbb{R})$. Using the mapping
$$x \to \frac{i - x}{i + x}$$
of $\mathbb{R}$ to the unit circle, show that:
a) The correspondence $U : \mathcal{H}_1 \to \mathcal{H}_2$ given by
$$U[F] = \frac{1}{\pi^{1/2}(i + x)} F(\frac{i - x}{i + x})$$

is a unitary mapping.
b) As a result show that

$$\left\{ \frac{1}{\pi^{1/2}(i+x)} \left(\frac{i-x}{i+x} \right)^n \right\}_{n=-\infty}^{\infty}$$

is an orthonormal basis of $\mathbb{L}^2(\mathbb{R})$.

6. Prove that the operator $T : \mathbb{L}^2[0, \infty] \to \mathbb{L}^2[0, \infty]$

$$T[f](x) = \frac{1}{\pi} \int_0^\infty \frac{f(y)}{x+y}\, dy$$

is bounded operator with norm $\|T\| \leq 1$.

7. Suppose that the multiplication operator $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is defined via $Ae_n = \alpha_n e_n$ where $\{e_i\}_{i=1}^\infty$ are the standard coordinate vectors and $\alpha_n \in \mathbb{R}$. Then show that A is bounded if and only if $\sup_n |\alpha_n| \leq M$.

8. Suppose that $\mathcal{K} : \mathbb{L}^2([0,1]) \to \mathbb{L}^2([0,1])$ is defined by

$$\mathcal{K}[f] = \int_0^1 k(x,y) f(y)\, dy\,,$$

where $k(x,y) \in \mathbb{L}^2([0,1] \times [0,1])$. Show that $\mathcal{K}$ is a bounded linear operator.

9. Give two examples of linear subspaces of $\mathbb{L}^2(\mathbb{R})$ which are not closed and find their closure.

10. Suppose that P_1 and P_2 are orthogonal projections onto subspaces S_1 and S_2. Show that $P_2 P_1$ is an orthogonal projection if and only if P_1 and P_2 commute, i.e. $P_1 P_2 = P_2 P_1$ and in this case $P_2 P_1$ projects onto $S_2 \cap S_1$. Give an example of two projection operators which do not commute.

11. Let $\mathcal{H} = \mathbb{L}^2(\mathbb{R})$. Let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be the Fourier transform

$$\mathcal{F}[f](x) = \int_{-\infty}^{\infty} e^{i2\pi xy} f(y)\, dx\,.$$

Then it is well known that $\mathcal{F}$ is a unitary map with the inverse

$$\mathcal{F}^{-1}[f](x) = \int_{-\infty}^{\infty} e^{-i2\pi xy} f(y)\, dx\,.$$

Let $f * g$ denote the convolution operator

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y) g(y)\, dy$$

Further, it is also known that

$$\mathcal{F}[fg](x) = \mathcal{F}[f] * \mathcal{F}[g]\,,$$

and

$$\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g]\,.$$

(a) Let $\chi_A(x)$ denote the indicator function of the set A, i.e. $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise. Suppose $k_0 > 0$. Show that

$$\mathcal{F}[\chi_{[-k_0,k_0]}] = \frac{\sin(2\pi k_0 x)}{\pi x}$$

(b) Let $K(x) = \mathcal{F}[\chi_{[-k_0,k_0]}](x)$. Show that

$$\int_{-\infty}^{\infty} K(x-z)K(z-y)\,dz = K(x-y)\,.$$

(c) Let $\mathcal{K} : \mathbb{L}^2(\mathbb{R}) \to \mathbb{L}^2(\mathbb{R})$ denote the operator defined by

$$\mathcal{K}[f](x) = \int_{-\infty}^{\infty} K(x-y)f(y)\,dy\,.$$

Show that $\mathcal{K}$ is a bounded operator.

(d) Use part (b) to show that $\mathcal{K}$ is a projection operator in the following sense, $\mathcal{K}[\mathcal{K}[f]] = \mathcal{K}[f]$

(e) Let $\mathcal{H}_0 \subset \mathcal{H}$ denote the subspace defined by:

$$f \in \mathcal{H}_0 \quad \text{if} \quad \mathcal{F}[f](x) = 0 \quad \forall |x| > k_0\,.$$

Show that $\mathcal{H}_0$ is a closed linear subspace. $\mathcal{H}_0$ is the subspace of band-limited functions with band-limit k_0.

(f) Show that $\mathcal{K}$ is the projection operator onto $\mathcal{H}_0$.