Problem set 2

Due date: Feb 19

February 13, 2018

1. Suppose that T is a symmetric bounded operator. Then show that

$$\|T\| = \sup\{|(Tf,f)|, \quad \|f\| = 1\}\,.$$

Hint: You may assume the polarization identity

$$(Tf,g) = \frac{1}{4}[(T(f+g),f+g)-(T(f-g),f-g)+i(T(f+ig),f+ig)-i(T(f-ig),f-ig)]$$

2. Suppose that G is a compact set in $\mathbb{R}^n$. Suppose that

$$T[f](x) = \int_G K(x,y)f(y)dy\,,$$

where $K : G \times G \to \mathbb{R}$ is a continuous function for all $x, y \in G$ except for $x = y$. Furthermore, suppose that K satisfies

$$|K(x,y)| \leq \frac{C}{|x-y|^{\alpha}}\,,$$

where $\alpha > 0$. Find the range of values of α for which the operator $T : \mathbb{L}^2(G) \to \mathbb{L}^2(G)$ is compact. Hint: Integral operators with continuous kernels are compact, and the norm limit of compact operators is compact.

3. Consider the operator $T : \mathbb{L}^2([0,1]) \to \mathbb{L}^2([0,1])$ defined by

$$T[f](t) = t \cdot f(t)$$

   (a) Prove that T is a bounded linear operator with $T = T^*$, but that T is not compact
   (b) However, show that T has no eigenvectors

   The multiplication operator defined above is shown to have a critical role in the design of quadratures (see , for example).

4. Let $\mathcal{H}$ be a Hilbert space with basis $\{e_k\}_{k=1}^{\infty}$. Verify that the operator T defined by

$$T(e_k) = \frac{e_{k+1}}{k}\,,$$

is compact, but has no eigenvectors

5. Let $\mathcal{H}$ be a Hilbert space with basis $\{e_k\}_{k=1}^{\infty}$. Verify that the operator T defined by

$$T(e_k) = \lambda_k e_k \, ,$$

is compact if and only if $\lim_{k\to\infty} \lambda_k \to 0$.

6. Let $\sigma(T)$ denote the spectrum of a compact operator $T : \mathcal{H} \to \mathcal{H}$. Show that $\lambda \in \sigma(T)$ if and only if $\overline{\lambda} \in \sigma(T^*)$

7. Let K be a Hilbert-Schmidt kernel which is real and symmetric, i.e. $K : [0,1] \times [0,1] \to \mathbb{R}$ satisfies $K(x,y) = K(y,x)$ and $K \in \mathbb{L}^2([0,1] \times [0,1])$. Let $T : \mathbb{L}^2([0,1]) \to \mathbb{L}^2([0,1])$ be defined by

$$T[f](x) = \int_0^1 K(x,y) f(y) dy \, .$$

Let $\phi_k(x)$ be the eigenvectors (with eigenvalues λ_k) that diagonalize T. Then:

   (a) $\sum_k |\lambda_k|^2 < \infty$
   (b) $K(x,y) = \sum_{k=1}^{\infty} \lambda_k \phi_k(x) \phi_k(y)$
   (c) Suppose $\tilde{T}$ is an operator which is compact and symmetric. Then $\tilde{T}$ is of Hilbert-Schmidt type if and only if $\sum_n |\lambda_n|^2 < \infty$, where $\{\lambda_n\}$ are the eigenvalues of $\tilde{T}$ counted according to their multiplicities

8. Let $\mathcal{H}$ be a Hilbert space.

   (a) If $T_1, T_2 : \mathcal{H} \to \mathcal{H}$ are compact symmetric operators which commute, i.e. $(T_1 T_2 = T_2 T_1)$, show that they can be diagonalized simultaneously. In other words, there exists an orthonormal basis for $\mathcal{H}$ which consists of eigenvectors for both T_1 and T_2.
   (b) A linear operator on $\mathcal{H}$ is normal if $TT^* = T^*T$. Prove that if T is normal and compact, then T can be diagonalized.
   (c) If U is unitary, and $U = \lambda I - T$, where T is compact, then U can be diagonalized.

9. Fredholm theory for non-zero index operators. An operator R is called a regularizer of an operator K if R is bounded and $RK = I - A_\ell$ and $KR = I - A_r$, where A_ℓ, A_r are compact.

   (a) Suppose that $K : \mathcal{H} \to \mathcal{H}$, and R is a regularizer of K, then $\dim\{\mathcal{N}(K)\} < \infty$ and $\dim\{\mathcal{N}(R)\} < \infty$
   (b) If $RK = I - A$, where A is compact, show that $\phi - A\phi = Rf$ has a solution for every $f \in \mathcal{N}(K^*)^\perp$
   (c) Now further assume that $N(I - A) = \{0\}$. Suppose that $S = (I - A)^{-1}$. Show that $\text{Ran}\,((I - KSR)) \subset \mathcal{N}(R)$ and that $\text{Ran}\,((I - KSR)^*) \subset \mathcal{N}(K^*)$. Combine the previous result and these results to show that $\phi = SRf$ also satisfies $K\phi = f$ as long as $f \in N(\mathcal{K}^*)^\perp$.
   (d) (optional, no extra credit) Show that Ran(K) $= \mathcal{N}(K^*)^\perp$ for any operator K which has a regularizer