Problem set 5

Due date: Apr 9

April 2, 2018

If needed you may assume that the topology on vector spaces is generated by a family of seminorms if that helps.

1. Suppose that $\mathcal{X}$ is a topological vector space (TVS), and let $\mathcal{U}$ be the collection of all open sets containing the origin. PRove the following.
   a) If $U \in \mathcal{U}$, then there is a $V \in \mathcal{U}$ such that $V + V \subset U$.
   b) If $U \in \mathcal{U}$, then there is a V in $\mathcal{U}$ such that $V \subset U$ and $\alpha V \subset V$ for all $|\alpha| \leq 1$.

2. Suppose that $\mathcal{X}$ is a vector space whose topology is defined by a family of seminorms $\mathcal{P}$, i.e., every open neighborhood of x_0 is of the form

$$U_{x_0} = \cap_{j=1}^{n} \{x \in \mathcal{X} : p_j(x - x_0) < \varepsilon_j\},$$

   where the seminorms further satisfy

$$\cap_{p \in \mathcal{P}} \{x : p(x) = 0\} = (0).$$

   Then show that $\mathcal{X}$ is a topological vector space with this topology.

3. Let $\mathcal{X}$ be a TVS. Show: a) if $x_0 \in \mathcal{X}$, then the mapping $x \to x + x_0$ is a homeomorphism, i.e., a continuous function and continuous inverse; b) if $\alpha \in \mathcal{F}$, and $\alpha \neq .0$, the map $x \to \alpha x$ is a homeomorphism.

4. Show that the weak topology is the smallest topology on $\mathcal{X}$ such that each $x^* \in \mathcal{X}^*$ is continuous.

5. Show that the weak$-*$ topology is the smallest topology on $\mathcal{X}^*$ such that each $x \in \mathcal{X}$, $x^* \to x^*(x)$ is continuous.

6. If $\mathcal{H}$ is a Hilbert space and $\{h_n\}$ is a sequence in $\mathcal{H}$ such that $h_n \to h$ weakly, i.e. $(h_n, f) \to (h, f)$ as $n \to \infty$ for each $f \in \mathcal{H}$. Suppose further that $\|h_n\| \to \|h\|$, then show that $\|h_n - h\| \to 0$.

7. Suppose that $\mathcal{X}$ is an infinite-dimensional normed space. If $\mathcal{S} = \{x \in \mathcal{X} : \|x\| = 1\}$, then the weak closure of S is $\{x : \|x\| \leq 1\}$.

8. In an infinite dimensional vector space, show that a bounded set cannot be open in the weak topology.