Practice Problem set 1

January 29, 2018

1. If P is the orthogonal projection onto a closed linear subspace S, then show that $P^2 = P$ and $P^* = P$

2. Prove the converse of the above result, i.e. if $P^2 = P$ and $P^* = P$, then P is orthogonal projection onto some closed subspace of the Hilbert space

3. Show that the multiplication operator $Te_k = \alpha_k e_k$, where $\{e_k\}_{k=1}^{\infty}$ is an orthogonal basis for a Hilbert space $\mathcal{H}$ is compact if and only if $|\alpha_k| \to 0$ as $k \to \infty$

4. Suppose that $w(x)$ is a non-negative bounded function. Suppose $K(x, y)$ satisfies

$$\int_{\mathbb{R}} |K(x,y)| w(y) dy \leq A w(x) \quad \text{for almost every } x \in \mathbb{R}$$

$$\int_{\mathbb{R}} |K(x,y)| w(x) dx \leq A w(y) \quad \text{for almost every } y \in \mathbb{R}\,.$$

   Prove that the integral operator defined by $Tf = \int_{\mathbb{R}} K(x,y) f(y) dy$ is bounded on $\mathbb{L}^2(\mathbb{R})$ with $\|T\| \leq A$.

5. Show that if T_1 and T_2 are bounded operators then

$$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$$

6. Suppose $\mathcal{H}_0$ is a pre-Hilbert space and $A : \mathcal{H}_0 \to \mathcal{H}_0$ is a bounded operator. Suppose that $\mathcal{H}$ is the completion of $\mathcal{H}$, show that there exists a bounded operator $\tilde{A} : \mathcal{H} \to \mathcal{H}$, such that $\tilde{A}h = Ah$ for all $h \in \mathcal{H}_0$. The operator $\tilde{A}$ is referred to as the continuous extension of the operator A.