PROBLEM SET 1

DUE DATE: FEB 14

- **Sections 1.2 - 2.3**
- Questions are either directly from the text or a small variation of a problem in the text.
- Collaboration is okay, but final submission must be written individually. Mention all collaborators on your submission.
- The terms in the bracket indicate the problem number from the text.

Section 1.2
1) (Prob 3,6, Pg 10) Solve the following equations and sketch some of the characteristics for each case.
a) $(1+x)\,u_x + u_y = 0$
Soln:

$$u(x,y) = f\left((1+x)\,e^{-y}\right)$$

b) $\sqrt{1-x^2}u_x + u_y = 0$
Soln:

$$u(x,y) = f\left(y - \arcsin(x)\right)$$

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2) (Prob 11, Pg 10) Solve $au_x + bu_y = f(x,y)$ where $f(x,y)$ is aa given function and a, b are constants with $a \neq 0$. Express the solution in the form

$$u(x,y) = \frac{1}{\sqrt{a^2+b^2}} \int_L f\,ds + g(bx - ay)$$

where g is an arbitrary function of one variable, L is the characteristic line segment from the y axis to the point (x,y) and the integral is a line integral. (Hint: Use the coordinate method.)

Solution:

The differential equation can be rewritten as

$$D_v u = \frac{1}{\sqrt{a^2+b^2}} f(x,y)$$

where $D_v u$ is the directional derivative of u in the unit direction $\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$. Integrating the above expression from $\left(0, y - \frac{a}{b}x\right)$ to (x,y), i.e. along, the characteristic, we get

$$u(x,y) = \frac{1}{\sqrt{a^2+b^2}} \int_L f\,ds + u\left(0, y - \frac{b}{a}x\right) . \tag{1}$$

Relabelling $g(-at) = u(0,t)$, we get the result.

Bonus: Where was the assumption $a \neq 0$ used in the above problem.

Solution: Clearly there is also a problem in equation 1 if $a = 0$. The issue is that characteristics run parallel to the y axis and the characteristic starting from (x,y) would not intersect the y axis.

Section 1.3
3) (Prob 6, Pg 19) Consider the heat equation in a long cylinder where the temperature only depends on t and the distance r to the axis of the cylinder. Here $r = \sqrt{x^2+y^2}$ is the cylinder coordinate. From the three dimensional heat equation derive the equation

$$u_t = k\left(u_{rr} + \frac{u_r}{r}\right) .$$

Solution:

The cylindrical coordinates are given by

$$\begin{aligned}x &= r\cos(\theta)\\ y &= r\sin(\theta)\\ z &= z\end{aligned}$$

and $r=\sqrt{x^2+y^2}$, $\theta=\arctan\left(\frac{y}{x}\right)$, $z=z$.
We are looking for solutions of the form $u(x,y,z,t)=u(r,t)$.

$$\begin{aligned}\partial_x u(r,t) &= \partial_r u(r,t)\frac{\partial r}{\partial x} = \partial_r u(r,t)\cdot\frac{x}{r}\\ \partial_{xx}u(r,t) &= \partial_r u(r,t)\cdot\partial_r\left(\frac{x}{r}\right)+\partial_x\partial_r u(r,t)\cdot\frac{x}{r} = \partial_r u(r,t)\frac{y^2}{r^3}+\partial_{rr}u(r,t)\cdot\frac{x^2}{r^2}.\end{aligned}$$

Similarly,

$$\partial_{yy}u(r,t)=\partial_r u(r,t)\frac{x^2}{r^3}+\partial_{rr}u(r,t)\cdot\frac{y^2}{r^2}.$$

And finally,

$$\partial_{zz}u(r,t)=0$$

The heat equation in cylindrical coordinates is then given by

$$\begin{aligned}u_t &= k\left(u_{xx}+u_{yy}+u_{zz}\right)\\ u_t &= k\left(\partial_r u(r,t)\frac{y^2}{r^3}+\partial_{rr}u(r,t)\cdot\frac{x^2}{r^2}+\partial_r u(r,t)\frac{x^2}{r^3}+\partial_{rr}u(r,t)\cdot\frac{y^2}{r^2}+0\right)\\ &= k\left(\partial_{rr}u+\frac{\partial_r u}{r}\right)\end{aligned}$$

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4) (Prob 8, Pg 19) For the hydrogen atom, let $e(t)=\int|u(t,\boldsymbol{x})|^2\,d\boldsymbol{x}$. Show that if $e(0)=1$, then $e(t)=1$ for all t. (Hint: compute $e'(t)$. Keep in mind that u is complex valued. Assume that $|u(t,\boldsymbol{x})|=0$ for $|\boldsymbol{x}|>R(t)$ where $R(t)<\infty$.
Solution:
Let $u(t,\boldsymbol{x})=v+iw$. Then $|u|^2=v^2+w^2$.

$$\begin{aligned}e'(t) &= \frac{d}{dt}\int|u(t,\boldsymbol{x})|^2\,d\boldsymbol{x}\\ &= \int\frac{d}{dt}\left(v^2+w^2\right)d\boldsymbol{x}\quad\text{(Since the integral converges absolutely)}\\ &= \int\left(2vv_t+2ww_t\right)d\boldsymbol{x}\end{aligned}$$

$u(t,\boldsymbol{x})$ satisfies the Schrodinger equation. Thus,

$$\begin{aligned}ihu_t &= \frac{h^2}{2m}\Delta u+\frac{e^2}{r}u\\ ih\left(v_t+iw_t\right) &= \frac{h^2}{2m}\Delta v+\frac{e^2}{r}v+i\left(\frac{h^2}{2m}\Delta w+\frac{e^2}{r}w\right)\\ v_t &= \frac{h}{2m}\Delta w+\frac{e^2}{rh}w\\ w_t &= -\left(\frac{h}{2m}\Delta v+\frac{e^2}{rh}v\right)\end{aligned}$$

$$\begin{aligned}e'(t) &= \int\left(2vv_t+2ww_t\right)d\boldsymbol{x}\\ &= C\int\left(v\Delta w-w\Delta v\right)d\boldsymbol{x}\end{aligned}$$

Consider the vector field $v\nabla w$ on the domain $r\le 2R(t)$. Then by the divergence theorem, we get

$$\int_{\mathbb{R}^3} \nabla \cdot (v \nabla w)\, d\boldsymbol{x} = \int_{|r| \leq 2R(t)} \nabla \cdot (v \nabla w)\, d\boldsymbol{x} = \int_{\partial(|r| \leq 2R)} v \frac{\partial w}{\partial n} dS\,,$$

where $\partial\left(|r| \leq 2R\right)$ is the boundary of the sphere. But $v = 0$ on $\partial\left(|r| \leq 2R\right)$. Thus,

$$\int_{\mathbb{R}^3} \nabla \cdot (v \nabla w)\, d\boldsymbol{x} = \int_{\mathbb{R}^3} \nabla v \cdot \nabla w\, d\boldsymbol{x} + \int_{\mathbb{R}^3} v \Delta w\, d\boldsymbol{x} = 0$$

$$\therefore \int_{\mathbb{R}^3} v \Delta w d\boldsymbol{x} = -\int_{\mathbb{R}^3} \nabla v \cdot \nabla w\, d\boldsymbol{x}\,.$$

From symmetry,

$$\therefore \int_{\mathbb{R}^3} w \Delta v\, d\boldsymbol{x} = -\int_{\mathbb{R}^3} \nabla w \cdot \nabla v\, d\boldsymbol{x}\,.$$

Therefore,

$$e'\left(t\right) = 0\,.$$

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5) (Prob 11, Pg 20) If $\nabla \times \boldsymbol{v} = \boldsymbol{0}$ in all of $\mathbb{R}^3$. Show that there exists a scalar function $\phi\left(x, y, z\right)$ such that $\boldsymbol{v} = \nabla\phi$.
Solution:
Let's construct the solution backwards. If there exists such a function ϕ such that $\boldsymbol{v} = \nabla\phi$, then firstly, we can change ϕ by any constant. So without loss of generality, $\phi\left(0,0,0\right) = 0$. Then to obtain the value at (x, y, z), we integrate ϕ along the path $(0,0,0) \to (x,0,0) \to (x,y,0) \to (x,y,z)$. Then

$$\phi\left(x,y,z\right) - \phi\left(0,0,0\right) = \int_0^x \partial_x \phi\left(t,0,0\right) dt + \int_0^y \partial_y \phi\left(x,t,0\right) dt + \int_0^z \partial_z \phi\left(x,y,t\right) dt$$
$$\phi\left(x,y,z\right) = \int_0^x v_1\left(t,0,0\right) dt + \int_0^y v_2\left(x,t,0\right) dt + \int_0^z v_3\left(x,y,t\right) dt$$

Now the only thing we need to verify is that if ϕ is as defined above, then is $\boldsymbol{v} = \nabla\phi$.

$$\partial_x \phi\left(x,y,z\right) = v_1\left(x,0,0\right) + \int_0^y \partial_x v_2\left(x,t,0\right) dt + \int_0^z \partial_x v_3\left(x,y,t\right) dt\,.$$

Since $\nabla \times \boldsymbol{v} = \boldsymbol{0}$, we have that

$$\partial_y v_1 = \partial_x v_2\,,$$
$$\partial_z v_1 = \partial_x v_3\,,$$
$$\partial_z v_2 = \partial_y v_3\,.$$

Plugging that back into the equation above, we get that,

$$\begin{aligned}\partial_x \phi\left(x,y,z\right) &= v_1\left(x,0,0\right) + \int_0^y \partial_y v_1\left(x,t,0\right) dt + \int_0^z \partial_z v_1\left(x,y,t\right) dt \\ &= v_1\left(x,0,0\right) + v_1\left(x,y,0\right) - v_1\left(x,0,0\right) + v_1\left(x,y,z\right) - v_1\left(x,y,0\right) \\ &= v_1\left(x,y,z\right)\end{aligned}$$

Similarly, it can be shown that $\partial_y \phi = v_2$ and $\partial_z \phi = v_3$.
Bonus: Is it true if $\nabla \times \boldsymbol{v} = \boldsymbol{0}$ on an arbirtrary domain D? Under what conditions on the domain D is it true?
No. True on simply connected domains!

Section 1.4
6) (Prob 6, Pg 25) Two homogeneous rods have the same cross section, specific heat c, and density ρ but different heat conductivities κ_1 and κ_2 and lengths L_1 and L_2. Let $k_j = \kappa_j / \left(c\rho\right)$ be their diffusion constants. They are welded together so that the temperature u and the flux κu_x are continuous. The left hand rod has its left end maintained at temperature 0. The right had rod has its right end at temperature T degrees.
a) Find the equilibrium temperature distribution in the composite rod.
b) Sketch it as a function of x in case $k_1 = 2$, $k_1 = 1$, $L_1 = 3$, $L_2 = 2$, $T = 10$.

Solutions:

Since u_1 and u_2 satisfy the steady state heat equation in $1D$, they satisfy $\partial_{xx}u_1 = 0$ and $\partial_{xx}u_2 = 0$. Thus, $u_1 = ax + b$ and $u_2 = cx + d$. The boundary conditions for u_1 and u_2 are

$$\begin{aligned} u_1(0) = 0 &\implies b = 0\,, \implies u_1(x) = ax \\ u_2(L_1 + L_2) = T &\implies u_2(x) = c(x - L_1 - L_2) + T \\ u_1(L_1) = u_2(L_1) &\implies aL_1 = -cL_2 + T \\ k_1 u'(L_1) = k_2 u_2'(L_1) &\implies k_1 a = k_2 c \end{aligned}$$

Solving the above system of equations for a, c we get

$$\begin{aligned} u_1(x) &= \frac{Tk_2}{L_1 k_2 + k_1 L_2} x = \frac{10x}{7} \\ u_2(x) &= \frac{Tk_1}{L_1 k_2 + L_2 k_1}(x - L_1 - L_2) + T = \frac{10(2x-3)}{7} \end{aligned}$$

Section 1.5

7) (Prob 1, Pg 27)Consider the boundary value ordinary differential equation

$$u''(x) + u(x) = 0\,, \quad u(0) = 0, u(L) = 0\,.$$

Clearly, the function $u(x) \equiv 0$ is a solution. Is the solution unique? Does the answer depend on L?

Solution: The solution is unique as long as $L \neq n\pi$. If $L = n\pi$, then $u(x) = \sin(x)$ is also a solution to the differential equation.

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8) (Prob 4, Pg 28) Consider the Neumann problem

$$\begin{aligned} \Delta u &= f(x, y, z) \quad \text{in D} \\ \frac{\partial u}{\partial \boldsymbol{n}} &= 0 \quad \text{on } \partial D \end{aligned}$$

a) Is the solution unique? What can we surely add to any solution to get another solution?

Solution: No, we can add a constant.

b) Use the divergence theorem and the PDE to show that

$$\int\int\int_D f(x, y, z)\, dx\, dy\, dz = 0$$

Solution:

$$\begin{aligned} \int\int\int_D f(x, y, z)\, dx\, dy\, dz &= \int\int\int_D \Delta u(x, y, z)\, dx\, dy\, dz \\ &= \int\int\int_D \nabla \cdot \nabla u(x, y, z)\, dx\, dy\, dz \\ &= \int\int_{\partial D} \nabla u(x, y, z) \cdot \boldsymbol{n}\, dS \\ &= \int\int_{\partial D} \frac{\partial u}{\partial \boldsymbol{n}}\, dS = 0 \end{aligned}$$

c) Give a physical interpretation of part a or part b either for heat flow or diffusion?

Solution: Since heat flux boundary conditions are specified, the temperature is well defined only up to a constant, changing the temperature everywhere by a constant does not change the heat flux through the boundary.

Section 2.1

9) (Prob 1, Pg 38) Solve $u_{tt} = 4u_{xx}$, $u(x, 0) = e^x$, $u_t(x, 0) = \sin(x)$.

Solution:

$$u(x, t) = e^x \cosh(2t) + \frac{1}{2}\sin(x)\sin(2t)$$

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10) (Prob 5, Pg 38) The hammer blow! A model for a note being played on a piano is the following.

$$u_{tt} = c^2 u_{xx} \quad u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x) \ .$$

Let $\phi(x) \equiv 0$, and $\psi(x) = 1$ for $|x| \leq a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile $u(x)$ at each of the time $t = a/2c$, a/c, $3a/2c$, $2a/c$, $5a/c$.

Solution:

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \mathbb{I}_{[-a,a]}(s)\, ds$$

where $\mathbb{I}_A(x)$ is the indicator function of the set A which is 1 if $x \in A$ and 0 otherwise. Thus,

$$u(x,t) = \frac{1}{2c} L([x-ct, x+ct] \cap [-a,a])$$

At $t = \frac{a}{2c}$

$$u(x,t) = \begin{cases} 0 & |x| \geq \frac{3a}{2} \\ a\left(1 - \frac{2|x|}{3a}\right) & |x| \leq \frac{3a}{2} \end{cases}$$

At $t = \frac{a}{c}$

$$u(x,t) = \begin{cases} 0 & |x| \geq 2a \\ 2a\left(1 - \frac{|x|}{2a}\right) & |x| \leq 2a \end{cases}$$

At $t = \frac{(m+1)a}{c}$

$$u(x,t) = \begin{cases} 0 & |x| \geq (m+2)\,a \\ 2a & |x| \leq ma \\ ((m+2)\,a - |x|) & ma < |x| < (m+2)\,a \end{cases}$$

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11) (Prob 8, Pg 38) A spherical wave is a solution of the three-dimensional wave equation of the form $u(r,t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right) \quad \text{("spherical wave equation")}$$

a) Change variables $v = ru$ to get the equation for $v : v_{tt} = c^2 v_{rr}$.
b) Solve for v given initial condition $u(r,0) = \phi(r)$ and $u_t(r,0) = \psi(r)$ where both $\phi(r)$ and $\psi(r)$ are even functions.

Solution:

$$\partial_t v = r\partial_t u \quad \partial_{tt} v = r\partial_{tt} u$$

$$\begin{aligned} \partial_r v &= r\partial_r u + \partial_r u \\ \partial_{rr} v &= r\partial_{rr} u + 2\partial_r u \\ \frac{1}{r}\partial_{rr} v &= \partial_{rr} u + \frac{2}{r}\partial_r u \end{aligned}$$

Thus, v satisfies the wave equation:

$$\partial_{tt} v = c^2 \partial_{rr} v$$

$$v(r,t) = \frac{1}{2}\left((r+ct)\,\phi(r+ct) + (r-ct)\,\phi(r-ct)\right) + \frac{1}{2c}\int_{r-ct}^{r+ct} s\psi(s)\, ds$$

$$u(r,t) = \frac{1}{2r}\left((r+ct)\,\phi(r+ct) + (r-ct)\,\phi(r-ct)\right) + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\psi(s)\, ds$$

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12) (Prob 9, Pg 38) Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = e^x$. (Hint: Factor the operator)
Solution:

$$u(x,t) = \frac{4}{5}\left(e^{x+\frac{t}{4}} - e^{x-t} + x^2 + \frac{1}{4}t^2\right).$$

Section 2.2
13) (Prob 5, Pg 41) Consider the damped string,

$$u_{tt} = c^2 u_{xx} - r u_t$$

Show that the energy decreases as a function of time. Prove uniqueness for the damped string.
Solution:

$$\begin{aligned} E(t) &= \frac{1}{2}\rho \int_{-\infty}^{\infty} u_t^2\, dx + \frac{1}{2} T \int_{-\infty}^{\infty} u_x^2\, dx \\ \frac{d}{dt} E(t) &= \int_{-\infty}^{\infty} u_t \left(\rho u_{tt} - T u_{xx}\right) dx \\ &= \rho \int_{-\infty}^{\infty} u_t \left(u_{tt} - c^2 u_{xx}\right) dx = -\rho r \int_{-\infty}^{\infty} u_t^2\, dx \le 0 \end{aligned}$$

Thus, the energy is a decreasing function of time. If $E(0) = 0$, then since $\frac{d}{dt}E(t) \le 0$ and $E(t) \ge 0$, we conclude that $E(t) \equiv 0$, which gives us that $\partial_t u \equiv 0$ and $\partial_x u \equiv 0$ from which uniqueness follows.

Section 2.3
14) (Prob 1, Pg 45) Consider the solution $1 - x^2 - 2kt$ of the diffusion equation. Find the locations of its maximum and mimum in the closed rectangle $\{0 \le x \le 1\,, \quad 0 \le t \le T\}$.

Solution: Due to maximum principle, we need to look for maximium or minimum only on the boundary. The maximum is at $x, t = (0,0)$ and the minimum is at $(x,t) = (1,T)$.

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15) (Prob 5, Pg 46) Consider the variable coefficient heat equation $u_t = x u_{xx}$

a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \le x \le 2\,, 0 \le t \le 1\}$. Note that the maximum is not achieved on the boundary.

Solution: The location of the maximum is at $(-1,1)$.

b) Where precisely does our proof of the maximum principle break down for this equation?

Solution: The sign of $x u_{xx}$ depends on x.