PROBLEM SET 2

DUE DATE: FEB 28

- **Sections 2.4-3.5**
- Questions are either directly from the text or a small variation of a problem in the text.
- Collaboration is okay, but final submission must be written individually. Mention all collaborators on your submission.
- The terms in the bracket indicate the problem number from the text.

Section 2.4

1) (Prob 9, Pg 53) Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x,0) = x^2$ by the following method.

i) Show that u_{xxx} also satisfies the diffusion equation with zero initial condition.

ii) Conclude that $u_{xxx}(x,t) \equiv 0$ for all (x,t) assuming $|u(x,t)| \leq Me^{ax^2}$ for all x, t. (Refer to practice problem set 2)

iii) Using ii, show that

$$u(x,t) = A(t)x^2 + B(t)x + C(t)$$

iv) Plug in the above expression for $u(x,t)$ in the differential equation, to obtain a system of differential equations for $A(t), B(t), C(t)$. (Hint: the functions $1, x, x^2$ are linearly independent)

v) Using the initial conditions, obtain initial values for $A(0), B(0)$ and $C(0)$ and solve the above system of differential equations to compute $u(x,t)$

2) (Prob 15, Pg 53) Using the energy method, prove uniqueness of the diffusion problem with Neumann boundary conditions:

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \quad 0 < x < 1, t > 0 \\ u(x,0) &= \phi(x) \\ u_x(0,t) &= g(t) \\ u_x(0,1) &= h(t) \end{aligned}$$

3) (Prob 16, Pg 54) Solve the diffusion equation with constant dissipation:

$$\begin{aligned} u_t - ku_{xx} + bu &= 0 \quad -\infty < x < \infty \\ u(x,0) &= \phi(x), \end{aligned}$$

where $b > 0$ is a constant by setting $u(x,t) = e^{-bt}v(x,t)$.

Section 3.1

4) (Prob 2, Pg 60) Solve $u_t = ku_{xx}$; $u(x,0) = 0$; $u(0,t) = 1$ on the half life $0 < x < \infty$. (Hint: Set $v(x,t) = u(x,t) - 1$.)

Section 3.2

5) (Prob 3, Pg 66) A wave $f(x+ct)$ travels along a semi-infinite string $0 < x < \infty$ for $t > 0$. Find the solution $u(x,t)$ of the string for $t > 0$ if the end $x = 0$ is fixed, i.e. $u(0,t) = 0$. Plot the solution in the three different regimes. Repeat the same exercise if $u_x(0,t) = 0$. Comment on the results.

6) (Prob 5, Pg 66) Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty$, $u(0,t) = 0$, $u(x,0) = 1$ and $u_t(x,0) = 0$ using the reflection method. Find the location of the singularity of the solution in the (x,t) space.

Section 3.3

7) (Prob 2, Pg 71) Solve the completely inhomogeneous diffusion problem on the half line

$$\begin{aligned} v_t - kv_{xx} &= f(x,t) \quad 0 < x < \infty, \quad 0 < t < \infty \\ v(0,t) &= h(t), \quad v(x,0) = \phi(x), \end{aligned}$$

by setting $V(x,t) = v(x,t) - h(t)$.

Section 3.4

8) (Prob 12,13, Pg 80) Derive the solution of the fully inhomogeneous wave equation on the half-line

$$v_{tt} - c^2 v_{xx} = f(x,t)\,, \quad 0 < x < \infty$$
$$v(x,0) = \phi(x) \quad v_t(x,0) = \psi(x)$$
$$v(0,t) = h(t)$$

by setting $V(x,t) = v(x,t) - h(t)$. Find the solution for $h(t) = t^2$, $\phi(x) = x$ and $\psi(x) = 0$.

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9) (Stability to small perturbations for the heat equation) Consider the inhomogeneous heat equation on the real line

$$u_t - ku_{xx} = f(x,t) \quad -\infty < x < \infty\,, \quad t > 0$$
$$u(x,0) = \phi(x)\ .$$

Show that the solutions are stable under small perturbations, i.e. Show that

$$\|u\|_{\mathbb{L}^\infty(\mathbb{R}\times[0,T])} = \sup_{x\in\mathbb{R},t\in[0,T]} |u(x,t)| \leq T\|f\|_{\mathbb{L}^\infty(\mathbb{R}\times[0,T])} + \|\phi\|_{\mathbb{L}^\infty(\mathbb{R})}.$$

Use the formula for the solution and the fact that

$$\int_{-\infty}^{\infty} S(x,t)\,dx = 1 \quad \forall t > 0\,.$$

Comment on why the above result implies stability of solutions to the data for the heat equation.