PROBLEM SET 3

DUE DATE: - MAR 9

- **Sections 4.1 - 4.3**
- Questions are either directly from the text or a small variation of a problem in the text.
- Collaboration is okay, but final submission must be written individually. Mention all collaborators on your submission.
- The terms in the bracket indicate the problem number from the text.

Section 4.1

1) (Prob 4,5 Pg 89) Consider waves in a resistant medium which satisfy the following PDE:

$$u_{tt} = c^2 u_{xx} - r u_t \quad 0 < x < \ell$$
$$u(0,t) = u(\ell,t) = 0 \quad \forall t > 0$$
$$u(x,0) = \phi(x) \quad 0 < x < \ell$$
$$\partial_t u(x,0) = \psi(x) \quad 0 < x < \ell\,,$$

where r is a constant. Write down a series expansion for the following cases:

i)

$$0 < r < \frac{2\pi c}{\ell}$$

$$u(x,t) = \sum_{n=1}^{\infty} e^{-rt}\left(A_n \cos(\beta_n t) + B_n \sin(\beta_n t)\right) \sin\left(\frac{n\pi x}{\ell}\right)\,,$$

where

$$\beta_n^2 = \frac{4n^2\pi^2c^2}{\ell^2} - r^2\,.$$

Here, the initial data satisfy,

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$
$$\psi(x) = \sum_{n=1}^{\infty} (-rA_n + B_n\beta_n) \sin\left(\frac{n\pi x}{\ell}\right)$$

ii)

$$\frac{2\pi c}{\ell} < r < \frac{4\pi c}{\ell}\,.$$

$$u(x,t) = \left(A_1 e^{-r_1 t} + B_1 e^{-r_2 t}\right) \sin\left(\frac{\pi x}{\ell}\right) + \sum_{n=2}^{\infty} e^{-rt}\left(A_n \cos(\beta_n t) + B_n \sin(\beta_n t)\right) \sin\left(\frac{n\pi x}{\ell}\right)\,,$$

where

$$\beta_n^2 = \frac{4n^2\pi^2c^2}{\ell^2} - r^2 \quad n \geq 2,$$

and

$$r_{1,2} = \frac{-r + \sqrt{r^2 - \frac{4\pi^2c^2}{\ell^2}}}{2}$$

Here, the initial data satisfy,

$$\phi(x) = (A_1 + B_1) \sin\left(\frac{\pi x}{\ell}\right) + \sum_{n=2}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$
$$\psi(x) = -(r_1 A_1 + r_2 B_1) \sin\left(\frac{\pi x}{\ell}\right) + \sum_{n=2}^{\infty} (-rA_n + B_n\beta_n) \sin\left(\frac{n\pi x}{\ell}\right)$$

Section 4.2

2) (Prob 2, Pg 92) Solve the wave equation with mixed boundary conditions using separation of variables, i.e. write down a series representation for the solution. You may assume that the initial conditions can be represented using an appropriate Fourier series.:

$$\begin{aligned} u_{tt} &= k u_{xx} \quad 0 < x < \ell \\ u_x(0,t) &= u(\ell,t) = 0 \\ u(x,0) &= \phi(x) \quad 0 < x < \ell \\ \partial_t u(x,0) &= \psi(x) \quad 0 < x < \ell \end{aligned}$$

Solution:

$$u(x,t) = \sum_{n=0}^{\infty} \left(A_n \cos\left(\frac{\left(n+\frac{1}{2}\right)\pi\sqrt{k}t}{\ell} \right) + B_n \sin\left(\frac{\left(n+\frac{1}{2}\right)\pi\sqrt{k}t}{\ell} \right) \right) \cos\left(\frac{\left(n+\frac{1}{2}\right)\pi x}{\ell} \right),$$

where

$$\begin{aligned} \phi(x) &= \sum_{n=0}^{\infty} A_n \cos\left(\frac{\left(n+\frac{1}{2}\right)\pi x}{\ell} \right), \\ \psi(x) &= \sum_{n=0}^{\infty} \frac{\left(n+\frac{1}{2}\right)\pi\sqrt{k}B_n}{\ell} \cos\left(\frac{\left(n+\frac{1}{2}\right)\pi x}{\ell} \right) \end{aligned}$$

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3) (Prob 3, Pg 92) Solve the Schrodinger equation $u_t = iku_{xx}$ for real k in the interval $0 < x < \ell$ with mixed boundary conditions $u_x(0,t) = u(\ell,t) = 0$.

Solution:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{\left(n+\frac{1}{2}\right)\pi x}{\ell} \right) e^{ik\frac{\left(n+\frac{1}{2}\right)^2\pi^2}{\ell^2}t}$$

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4) (Prob 4, Pg 92) (Periodic boundary conditions) Consider diffusion inside an enclosed circular tube. Let it's length be 2ℓ. Let x denote the arclength parameter. The concentration of the diffusing substance satisfies

$$\begin{aligned} u_t &= k u_{xx} \quad -\ell \le x \le \ell \\ u(-\ell,t) &= u(\ell,t) \\ \partial_x u(-\ell,t) &= \partial_x u(\ell,t), \end{aligned}$$

Show that the solution is given by

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi x}{\ell}\right) + B_n \sin\left(\frac{n\pi x}{\ell}\right) \right) \exp\left(-\frac{n^2\pi^2 kt}{\ell^2} \right).$$

Section 4.3

5) (Prob 2, Pg 100) Consider the Robin eigenvalue value problem

$$\begin{aligned} X'' &= -\lambda X \\ X'(0) - a_0 X(0) &= X'(\ell) + a_\ell X(\ell) = 0. \end{aligned}$$

a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_\ell = -a_0 a_\ell \ell$.
b) Find the eigenfunctions corresponding to the zero eigenvalue.

Solution:

The eigenfunction corresponding to the zero eigenvalue is

$$X(x) = \beta_0 + \beta_1 x.$$

On imposing the boundary conditions, we get

$$\begin{aligned}\beta_1 &= a_0\beta_0\\ \beta_1 &= -a_\ell\left(\beta_0+\beta_1\ell\right)\\ a_0\beta_0 &= -a_\ell\beta_0 - a_\ell a_0\beta_0\ell \implies a_0+a_\ell = -a_0a_\ell\ell\end{aligned}$$

The corresponding eigenfunction is

$$X\left(x\right) = \beta_0\left(1+a_0x\right)\,.$$

6) (Prob 4, Pg 100) Consider the Robin eigenvalue problem. If $a_0<0$, $a_\ell<0$ and $-a_0-a_\ell<a_0a_\ell\ell$, show that there are two negative eigenvalues. (Hint: Show that the rational curve

$$y = -\frac{\left(a_0+a_\ell\right)\gamma}{\gamma^2+a_0a_\ell}\,,$$

has a single maximum and crosses the line $y=1$ in two places. Deduce that it crosses the tanh curve in two places as well.

Solution: $\lambda=-\gamma^2$ if and only if $\gamma>0$ satisfies

$$\tanh\left(\gamma\ell\right) = -\frac{\left(a_0+a_\ell\right)\gamma}{\gamma^2+a_0a_\ell}$$

The location of the maximum of the function

$$y\left(\gamma\right) = \frac{-\left(a_0+a_\ell\right)\gamma}{\gamma^2+a_0a_\ell}$$

is at $\gamma_{max}=\sqrt{a_0a_\ell}$ and the maximum value is

$$y\left(\gamma_{max}\right) = -\frac{\left(a_0+a_\ell\right)}{2\sqrt{a_0a_\ell}}\,.$$

Since arithmatic mean of two numbers is greater than the geometric mean of two numbers, we conclude that

$$y\left(\gamma_{\max}\right)\geq 1\,.$$

Furthermore, we note that $\tanh\left(\gamma_{\max}\ell\right)<1$ and limit $\gamma\to\infty$ $\tanh\left(\gamma\ell\right)=1$ and $y\left(\gamma\right)=0$. Owing to the continuity of $\tanh\left(\gamma\ell\right)$ and $y\left(\gamma\right)$, we conclude that there exists a $\gamma_1>\gamma_{\max}$ such that $\tanh\left(\gamma_1\ell\right)=y\left(\gamma_1\right)$.

The second intersection of the two functions γ_2 satisfies $0<\gamma_2<\gamma_{\max}$. $y\left(0\right)=\tanh\left(0\cdot\ell\right)=0$. Furthermore, $y'\left(0\right)=-\frac{\left(a_0+a_\ell\right)}{a_0a_\ell}$ and $\frac{d}{d\gamma}\tanh\left(\gamma\ell\right)|_{\gamma=0}=\ell$. Thus, there exists δ_0, sufficiently small, such that $y\left(\delta\right)<\tanh\left(\delta\ell\right)$ and we know that $y\left(\gamma_{\max}\right)>\tanh\left(\gamma_{\max}\ell\right)$. Again by continuity, we conclude that there exists $0<\gamma_2<\gamma_{\max}$ such that $y\left(\gamma_2\right)=\tanh\left(\gamma_2\ell\right)$.

7) (Prob 18, Pg 102-103). A tuning fork may be regarded as a pair of vibrating flexible bars with a certain degree of stiffness. The governing equation for such a fork is given by the fourth order PDE

$$\begin{aligned}u_{tt}+c^2u_{xxxx} &= 0 \quad 0<x<\ell\\ u\left(0,t\right)=u_x\left(0,t\right) &= 0 \quad \text{(Fixed end/clamped boundary conditions)}\\ u_{xx}\left(\ell,t\right)=u_{xxx}\left(\ell,t\right) &= 0 \quad \text{(Free end/No stress at the end)}\,.\end{aligned}$$

a) Separate the time and space variables to get the eigenvalue problem

$$X'''' = \lambda X\,.$$

b) Show that 0 is not an eigenvalue.

Solution: The eigenfunction corresponding to

$$X'''' = 0$$

is given by $X\left(x\right)=a_0+a_1x+a_2x^2+a_3x^3$ where $X\left(x\right)$ satisfies the boundary conditions

$$X\left(0\right)=X'\left(0\right)=0=X''\left(\ell\right)=X'''\left(\ell\right)\,.$$

On imposing the boundary conditions, we get

$$\begin{aligned}a_0 &= X\left(0\right)=0\\ a_1 &= X'\left(0\right)=0\\ 2a_2+6a_3\ell &= X''\left(\ell\right)=0\\ 6a_3 &= X'''\left(\ell\right)=0\end{aligned}$$

From which we conclude that $X\left(x\right)=0$.

c) Assuming that all the eigenvalues are positive, write them as $\lambda=\beta^4$ and find the equation for β.

Solution: $\cosh\left(\beta\ell\right)\cos\left(\beta\ell\right)=-1$

d) Find the frequencies of vibration.

Solution: $\beta_1\ell = 1.88$, $\beta_2\ell = 4.69$, $\beta_3\ell = 7.85$ and the frequency of vibration is

$$\lambda_n = c\beta_n^2$$

e) Compare the answer in part (d) with the overtones of the vibrating string by comparing at the ratio β_2^2/β_1^2 . Explain why you hear an almost pure tone when you listen to a tuning fork.

Solution: For the bar $\frac{\lambda_2}{\lambda_1} = 6.27$ while that for a string is 2. Thus, relative to the fundamental frequency, the first overtone of the bar is higher than the fifth overtone.