PROBLEM SET 4

DUE DATE: - MAR 28

- **Chap 5**
- Questions are either directly from the text or a small variation of a problem in the text.
- Collaboration is okay, but final submission must be written individually. Mention all collaborators on your submission.
- The terms in the bracket indicate the problem number from the text.

Section 5.1

1) (Prob 2, Pg 111) Compute the fourier sine and fourier cosine series of $\phi(x) = x^2$ for $0 \le x \le 1$.

Solution:

$$x^2 = \frac{1}{6} + \sum_{m=1}^{\infty} \frac{4}{m^2\pi^2} (-1)^m \cos(m\pi x)$$
$$x^2 = 2\sum_{m=1}^{\infty} \frac{\left(2-\pi^2 m^2\right)(-1)^m - 2}{\pi^3 m^3} \sin(m\pi x)$$

2) (Prob 10, Pg 111) A string (of tension T and density ρ) with fixed ends at $x=0$ and $x=\ell$ is hit by a hammer so that $u(x,0)=0$ and $\partial_t u(x,0) = V$ for $x \in \left[-\delta + \frac{1}{2}\ell, \delta + \frac{1}{2}\ell\right]$ and $\partial_t u(x,0) = 0$ otherwise. Find the solution explicitly in series form. Find the energy

$$E_n[h](t) = \frac{1}{2}\int_0^{\ell} \left[\rho \partial_t h(x,t)^2 + T\partial_x h(x,t)^2\right] dx$$

of the nth harmonic $h = h_n$. Conclude that if δ is small (a concentrated blow), each of the first few overtones has almost as much energy as the fundamental mode.

Solution:

$$E_n = \frac{4\ell\rho V^2}{(n\pi)^2} \sin^2\left(\frac{n\pi\delta}{\ell}\right) \approx 4\rho V^2 \ell^{-1}\delta^2$$

as long as $n\delta \ll 1$.

Section 5.3

3) (Prob 4, Pg 123) Consider the problem $u_t = k u_{xx}$ for $0 < x < \ell$, with the boundary conditions $u(0,t) = U$, $u_x(\ell,t) = 0$, and the initial condition $u(x,0) = 0$, where U is a constant.

a) Find the solution in series form (Consider $u(x,t) - U$)

b) Using a direct argument, show that the series converges for each time slice $t = t_0$ uniformly in x for $t_0 > 0$.

Solution:

a)

$$u(x,t) = U - \frac{4U}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2m+1)} e^{-\frac{\left(m+\frac{1}{2}\right)^2\pi^2 k t}{\ell^2}} \sin\left(\left(m+\frac{1}{2}\right)\frac{\pi x}{\ell}\right)$$

b) Uniform convergence for time slice t_0 follows from the fact that

$$\sum_{n=0}^{\infty} \frac{1}{(2m+1)} e^{-\frac{\left(m+\frac{1}{2}\right)^2\pi^2 k t_0}{\ell^2}} < \infty \tag{1}$$

4) (Prob 9, Pg 123) Show that the boundary conditions

$$X(b) = \alpha X(a) + \beta X'(a) \quad \text{and} \quad X'(b) = \gamma X(a) + \delta X'(a)$$

on an interval $a \le x \le b$ are symmetric if and only if $\alpha\delta - \beta\gamma = 1$.

Solution:
An operator is symmetric if

$$\begin{aligned}(u,Lv)-(Lu,v)=uv'-u'v|_a^b &= u(b)\,v'(b)-u'(b)\,v(b)-u(a)\,v'(a)+u'(a)\,v(a)\\ &= (\alpha u(a)+\beta u'(a))(\gamma v(a)+\delta v'(a))-(\gamma u(a)+\delta u'(a))(\alpha v(a)+\beta v'(a))-u(a)\,v'(a)+u'(a)\,v(a)\\ &= (\alpha\delta-\beta\gamma-1)\,u(a)\,v'(a)-(\alpha\delta-\beta\gamma-1)\,u'(a)\,v(a)=0 \iff \alpha\delta-\beta\gamma=1\end{aligned}$$

Section 5.4
5) (Prob 1, Pg 134) $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ is a geometric series.
a) Does it converge pointwise in the interval $-1<x<1$?
b) Does it converge uniformly in the interval $-1<x<1$?
c) Does it converge in the $\mathbb{L}^2$ sense in the interval $-1<x<1$?
Solution:
a) Yes, to the function $\frac{1}{1+x^2}$
b) No, the convergence is not uniform as

$$\sup_{|x|<1}\left|\sum_{n=0}^{N}(-1)^n x^{2n}-\frac{1}{1+x^2}\right|=\sup_{|x|<1}\left|\frac{-x^{2N+2}}{1+x^2}\right|=\frac{1}{2}\not\to 0$$

c) Yes, the function converges in $\mathbb{L}^2$

$$\begin{aligned}\lim_{n\to\infty}\int_{-1}^{1}\left|f_n-\frac{1}{1+x^2}\right|^2dx &= \int_{-1}^{1}\frac{x^{4N+4}}{(1+x^2)^2}dx\\ &\le \frac{1}{2(4N+5)}\to 0\end{aligned}$$

6) (Prob 3, Pg 134) Let $f_n(x)$ be the sequence of functions defined as follows: $f_n(0.5)=0$, $f_n(x)=\gamma_n$ in the interval $[0.5-\frac{1}{n},0.5)$, let $f_n(x)=-\gamma_n$ in the interval $(0.5,0.5+\frac{1}{n}]$ and let $f_n(x)=0$ elsewhere. Show that:
a) $f_n(x)\to 0$ pointwise.
b) The convergence is not uniform.
c) $f_n\to 0$ in the $\mathbb{L}^2$ sense if $\gamma_n=n^{1/3}$
d) f_n does not converge in the L^2 sense if $\gamma_n=n$.
Solution:
a) Let $x<0.5$, then for large enough $n>N$, $x\notin[0.5-\frac{1}{n},0.5)$, and $f_n(x)=0$. Similarly if $x>0.5$, then for large enough N, $x\notin(0.5,0.5+\frac{1}{n}]$ and $f_n(x)=0$. If $x=0.5$, $f_n(x)=0$ for all n. Thus $f_n\to 0$ pointwise
b)

$$\sup_{0<x<1}|f_n-0|=\gamma_n$$

and the convergence is not uniform unless $\gamma_n\to 0$.
c,d)

$$\int_0^1|f_n-0|^2\,dx=\int_{0.5-\frac{1}{n}}^{0.5+\frac{1}{n}}\gamma_n^2dx=\frac{2}{n}\gamma_n^2$$

From the above expression, if $\gamma_n=n^{\frac{1}{3}}$, then $f_n\to 0$ in $\mathbb{L}^2$. If $\gamma_n=n$, then $f_n\not\to 0$ in $\mathbb{L}^2$.