PROBLEM SET 6

DUE DATE: - APR 25

- **Chap 7, 9.1**
- Questions are either directly from the text or a small variation of a problem in the text.
- Collaboration is okay, but final submission must be written individually. Mention all collaborators on your submission.
- The terms in the bracket indicate the problem number from the text.

Section 7.1

1) (Prob 5, Pg 184) Prove Dirichlet's Principle for Neumann boundary condition. It asserts that among all real-valued functions $w(\boldsymbol{x})$ on D, the quantity

$$E[w] = \frac{1}{2}\iiint_D |\nabla w|^2 \,, d\boldsymbol{x} - \iint_{\partial D} hw \, dS \,,$$

is the smallest for $w = u$, where u is the solution of the Neumann problem

$$-\Delta u = 0 \quad \text{in } D \quad \frac{\partial u}{\partial n} = h(\boldsymbol{x}) \quad \text{on } \partial D \,,$$

where h satisfies the constraint

$$\iint_{\partial D} h(\boldsymbol{x}) \, dS = 0 \,.$$

Note that there are no restrictions on w as opposed to the Dirichlet principle for Dirichlet boundary conditions, the function $h(\boldsymbol{x})$ appears in the energy and the energy does not change if you add a constant to w. Comment on the last bit in context of solutions to the Neumann problem for Laplace's equation.

Section 7.2

2) (Prob 1, Pg 187) Derive the representation formula for hamronic functions in two dimensions

$$u(\boldsymbol{x}_0) = \frac{1}{2\pi}\int_{\partial D} \left[u(\boldsymbol{x}) \frac{\partial}{\partial n} (\log|\boldsymbol{x} - \boldsymbol{x}_0|) - \frac{\partial u}{\partial n} \log|\boldsymbol{x} - \boldsymbol{x}_0| \right] ds$$

Section 7.3

3) (Prob 1, Pg 190) Show that the Green's function is unique. (Hint: Take the difference of two of them)

Section 7.4

4) (Prob 7,8 Pg 196) a) If $u(x, y) = f\left(\frac{x}{y}\right)$ is a harmonic function, solve the ODE satisfied by f.
b) Show that $\partial_r u \equiv 0$, where $r = \sqrt{x^2 + y^2}$
c) Suppose $v(x, y)$ is any $\{y > 0\}$ such that $\partial_r v \equiv 0$. Show that $v(x, y)$ is a function of the quotient $\frac{x}{y}$.
d) Find the boundary values $\lim_{y \to 0} u(x, y) = h(x)$
e) Find the harmonic function in the half plane $\{y > 0\}$ with boundary data $h(x) = 1$ for $x > 0$ and $h(x) = 0$ for $x < 0$.
f) Find the harmonic function in the half plane $\{y > 0\}$ with boundary data $h(x) = 1$ for $x > a$ and $h(x) = 0$ for $x < a$.
5) (Prob 17, Pg 197) a) Find the Green's function for the quadrant

$$Q = \{(x, y) : x > 0 \,, y > 0\} \,.$$

b) Use the answer in part (a) to solve the Dirichlet problem

$$\begin{aligned} \Delta u = 0 \quad &\text{in } Q \\ u(0, y) = g(y) \quad &y > 0 \\ u(x, 0) = h(x) \quad &x > 0. \end{aligned}$$

6) (Prob 21, Pg 198) The Neumann function $N(\boldsymbol{x}, \boldsymbol{y})$ for a domain D is defined exactly like the Green's function with the following conditions:

$$N(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{4\pi |x - y|} + H(\boldsymbol{x}, \boldsymbol{y})$$

where $H(\boldsymbol{x}, \boldsymbol{y})$ is a harmonic function of $\boldsymbol{x}$ for each fixed $\boldsymbol{y}$, and

$$\frac{\partial N}{\partial n} = c \quad \boldsymbol{x} \in \partial D$$

for a suitable constant c.

a) Show that $c = \frac{1}{A}$ where A is the area of the boundary ∂D.

b) State and prove the analog of Theorem 7.3.1, expressing the solution of the Neumann problem in terms of the Neumann function.

7) (Prob 22: Pg 198) Solve the Neumann problem in the half plane:

$$\Delta u = 0 \quad \text{in } \{y > 0\}, \frac{\partial u}{\partial y}(x, 0) = h(x)$$

and $u(x, y)$ is bounded at ∞.

Section 9.1

8) (Prob 1, Pg 233) Find all three-dimensional plane waves: i.e., all the solutions of the wave equation of the form $u(\boldsymbol{x}, t) = f(\boldsymbol{k} \cdot \boldsymbol{x} - ct)$ where $\boldsymbol{k}$ is a fixed vector and f is a function of one variable

9) (Prob 8, Pg 234) Consider the equation

$$\partial_{tt} u - c^2 \Delta u + m^2 u = 0,$$

where $m > 0$, known as the Klein-Gordon equation.

a) What is the energy? Show that it is a constant.

b) Prove the causality principle for this equation.