PRACTICE PROBLEM SET 4

- **Chap 5**
- Questions are either directly from the text or a small variation of a problem in the text.
- Collaboration is okay, but final submission must be written individually. Mention all collaborators on your submission.
- The terms in the bracket indicate the problem number from the text.

Section 6.1

1) Maxwell's equations in frequency domain reduce to computing solutions of Helmholtz equation given by

$$\Delta u + k^2 u = 0$$

Find all radially symmetric solutions to the above PDE in three dimensions.

(Note the sign of the laplacian has changed)

$$\Delta_r = \partial_{rr} + \frac{2}{r}\partial_r .$$

Thus, radially symmetric solutions to the PDE $u(r)$, satisfy the ODE

$$u'' + \frac{2}{r}u' + k^2 u = 0 .$$

If we set $v = ru$. Then v satisfies the ODE

$$v'' + k^2 v = 0 .$$

Thus

$$v(r) = e^{\pm ikr} ,$$

and

$$u(r) = \frac{e^{\pm ikr}}{r} .$$

Section 6.2

2) (Prob 4, Pg 165) Find the harmonic function in the square $\{0 < x < 1, 0 < y < 1\}$ with the boundary conditions, $u(x,0) = x$, $u(x,1) = 0$, $u_x(0,y) = 0$ and $u_x(1,y) = y^2$.

Solution:

Let us construct the solution in two phases: The first corresponding to the data on the x boundaries. A solution that takes care of the initial conditions is given by

$$u_1(x,y) = x(1-y) .$$

Clearly u is harmonic and satisfies $u_1(x,0) = x$ and $u_1(x,1) = 0$. Then we need to seek a solution

$$u_2(x,0) = 0 , \quad u_2(x,1) = 0 , \quad \partial_x u_2(0,y) = -(1-y) , \quad \partial_x u_2(1,y) = y^2 + y - 1 .$$

And now we can use a separation of variable ansatz to obtain the solution in 1 shot.

$$u_2(x,y) = \sum_{n=1}^{\infty} \sin(n\pi y)\left(A_n \sinh(n\pi x) + B_n \cosh(n\pi x)\right)$$

$$\partial_x u_2(x,y) = \sum n\pi \cos(n\pi y)\left(A_n \sinh(n\pi x) + B_n \cosh(n\pi x)\right)$$

where

$$y - 1 = \sum_{n=1}^{\infty} n\pi B_n \cos(n\pi y)$$

and

$$y^2 + y - 1 = \sum_{n=1}^{\infty} n\pi \left(B_n \cosh(n\pi) + A_n \sinh(n\pi)\right) \cos(n\pi x) \,.$$

Section 6.3

3) (Prob 1, Pg 172)Suppose u is harmonic in the disc $D = \{r < 2\}$ and that $u = 3\sin(2\theta) + 1$ for $r = 2$. Without computing the solution, answer the following:

a) Find the maximum value of u in $\overline{D}$

b) What is the value of u at the origin.

Solution:

Max value is 4. Since maximum is achieved on the boundary.

Value at the origin is 1 using mean value principle.

Section 6.4

4) (Prob 12 (b), Pg 177) Suppose D is a connected domain with a smooth boundary. A stronger version of the maximum principle is the following. Suppose u is not a constant, then u achieves it's maximum at $\boldsymbol{x}_0$ on the boundary and moreover

$$\frac{\partial u}{\partial n}(\boldsymbol{x}_0) > 0 \,.$$

Use this to show uniqueness of Neumann problems, up to constants.

Solution: The proof is fairly technical.

Section 7.1

5) (Prob 2, Pg 184) Prove uniqueness of the Neumann problem (up to constants) using the energy method.

$$\iint_D u \Delta u \, dV = \iint_{\partial D} u \frac{\partial u}{\partial n} dS - \iint_D |\nabla u|^2 \, dV \,.$$

Thus if u is harmonic and $\frac{\partial u}{\partial n}$ is 0 on the boundary, then

$$\iint_D |\nabla u|^2 \, dV = 0$$

and thus u is a constant.

Bonus question:

Mean value property $\implies$ smoothness.

i) Convince yourself that the bump function defined by

$$\phi(x) = \begin{cases} e^{-\left(\frac{1}{1-x^2}\right)} & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

ii) Show that

$$u(\boldsymbol{x}) = \int_{\mathbb{R}^2} \phi(|\boldsymbol{x} - \boldsymbol{y}|) \, \sigma(\boldsymbol{y}) \, dV$$

is a smooth function is σ is continuous.

iii) Suppose u satisfies the mean value property in $\mathbb{R}^2$, i.e.

$$u(\boldsymbol{x}) = \frac{1}{\pi R^2} \int_{B_r(\boldsymbol{x})} u(\boldsymbol{y}) \, dV_{\boldsymbol{y}}$$

then show that

$$u(\boldsymbol{x}) = C \int_{\mathbb{R}^2} \phi(|\boldsymbol{x} - \boldsymbol{y}|) \, u(\boldsymbol{y}) \, dV$$

for some constant C and thus u is smooth.

iv) If you are really brave, adapt this proof to show smoothness of harmonic functions in bounded domains. The issue is the boundary, but you can instead use an appropriately scaled version of $\phi\left(\frac{x}{\epsilon}\right)$ to get smoothness for all points whose distance from the boundary is at least ϵ.