Linear Algebra with Application.

HW5

Selected problems

§ 4.1 #4, #22

§ 4.2 #5, #13, #17, #25

§ 4.1

#4

- Let $A \in Mat_{m,n}(\mathbb{R})$, $B \in Mat_{n,l}(\mathbb{R})$ such that $AB = 0$

Note that columns of B are exactly $Be_1, \cdots, Be_l$ where

$e_j = (0, \cdots, 0, 1, 0, \cdots, 0)^t \quad (j = 1, \cdots, l)$

↑ j-th entry.

Now $AB = 0 \Rightarrow A(Be_j) = (AB)e_j = 0$

$\Rightarrow$ Be_j's are in the kernel of A.

Similarly, rows of A are exactly $f_1 A, \cdots, f_m A$ where

$f_i = (0, \cdots, 0, 1, 0, \cdots, 0) \quad (i = 1, \cdots, m)$

↑ i-th entry

Again, $AB = 0 \Rightarrow (f_i A)B = f_i(AB) = 0$

Hence $f_i A$'s are in the left null space of B.

- Suppose $A, B \in Mat_{3,3}(\mathbb{R})$, $AB = 0$ and $rk(A) = rk(B) = 2$.

By dimension thm, $Null(A) + rk(A) = 3$ so $Null(A) = 1$.

On the other hand, by previous observation, $Ker(A)$ already contains the column space of B, which is 2-dimensional since $rk(B) = 2$. And we get $Null(A) = \dim ker(A) \geq 2 \quad \rightarrow\leftarrow$.

#22

- $P = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0 \}$

$= \{ \vec{x} \in \mathbb{R}^4 \mid \vec{n} \cdot \vec{x} = 0 \}$ where $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{n} = (1, 1, 1, 1)$

Hence this is a hyperplane in $\mathbb{R}^4$ passing through the origin whose normal vector is $\vec{n}$. So $P^\perp = \langle (1, 1, 1, 1) \rangle$

- In order that a matrix A has P as its kernel, By Fundamental thm of Linear algebra (cf. p198 of textbook), row space of A should be $P^\perp = \langle (1, 1, 1, 1) \rangle$.

Examples: $(1, 1, 1, 1)$, $\begin{pmatrix} 2, 2, 2, 2 \\ 0, 0, 0, 0 \end{pmatrix}$, etc.

§ 4.2

#5

$a_1 = (-1, 2, 2)^t$

$a_1^t a_1 = (-1)^2 + 2^2 + 2^2 = 9$

$$a_1 a_1^t = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \cdot (-1, 2, 2) = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$$

$$\therefore P_1 = \frac{a_1 a_1^t}{a_1^t a_1} = \frac{1}{9} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$$

$a_2 = (2, 2, -1)^t$

$a_2{}^t a_2 = 2^2 + 2^2 + (-1)^2 = 9$

$$a_2 a_2{}^t = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \cdot (2, 2, -1) = \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$\therefore P_2 = \frac{a_2 a_2{}^t}{a_2{}^t a_2} = \frac{1}{9} \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$P_1 P_2 = 1/9^2 \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} = 0.$$

Interpretation. What is the map "$\vec{x} \longmapsto P_1 P_2 \vec{x}$" ?

Multiplying P_i on the left corresponds to projecting that vector on the "line generated by $\vec{a_i}$" $= \langle \vec{a_i} \rangle$ $(i=1,2)$

Observe that $\vec{a_1}, \vec{a_2}$ are orthogonal. ($\because \vec{a_1} \cdot \vec{a_2} = 0$)

Let $\vec{x} \in \mathbb{R}^3$. Then $P_2 \vec{x}$ is a projection of x to $\langle \vec{a_2} \rangle$. In particular, $P_2 \vec{x} \parallel \vec{a_2}$ and hence $P_2 \vec{x} \perp \vec{a_1}$. $\therefore P_1(P_2 x) = 0$ which explains why $P_1 P_2$ should be identically 0.

$\boxed{\#13}$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

i) We can use the formula directly to get

$$P = A \cdot (A^t A)^{-1} \cdot A^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right]^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} Id_{3\times3} & 0 \\ \hline 0 & 0 \end{array} \right)$$

and that $$Pb = \left(\begin{array}{c|c} Id_{3\times3} & 0 \\ \hline 0 & 1 \end{array} \right) \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}.$$

ii) Or, notice that in this case, the projection map we are looking for is easy to see, namely $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$

$(x, y, z, w) \longmapsto (x, y, z, 0)$

Which corresponds to $\left(\begin{array}{c|c} Id_{3\times 3} & 0 \\ \hline 0 & 0 \end{array}\right) \in Mat_{4,4}(\mathbb{R})$

#17 $P^2 = P$

$\Rightarrow (I-P)^2 = I - 2P + P^2 = I - 2P + P = I - P$.

I claim that $\operatorname{Ran}(I-P) = \operatorname{Ker} P$.

Proof. $(\subseteq)$ Let $y \in \operatorname{Ran}(I-P)$. Then $y = (I-P)x$ for some x.

$\Rightarrow Py = P(I-P)x = (P-P^2)x = 0 \quad (\because P = P^2)$

$\Rightarrow y \in \operatorname{Ker} P$

$(\supseteq)$ Let $x \in \operatorname{Ker} P$. Then $Px = 0 \Rightarrow x = x - Px = (I-P)x$

$\Rightarrow x \in \operatorname{Ran}(I-P)$. ▨ Claim.

Hence $I-P$ projects onto the Kernel of P.

#25 $rk(P) \leq n$ because $\operatorname{Ran} P$ is contained in n-dimensional subspace, let's call this S.

On the other hand, for $\forall v \in S$, $Pv = v$ so $\operatorname{Ran} P \supseteq S$

Which means $rk(P) \geq n$, Hence P has rank n.

[S is the image (range) of P]