HW 7.

Complete solution.

Math 222

Fall 2016.

§ 4.3

$\boxed{\#1}$ To minimize $\left\| \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} - \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} \right\|$

We need to solve $A^t A \tilde{x} = A^t b$ (where $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$)

$$A^t A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 8 & 26 \end{pmatrix}$$

$$A^t b = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 36 \\ 112 \end{pmatrix}$$

Solving $A^t A \tilde{x} = A^t b$:

$$\left(\begin{array}{cc|c} 4 & 8 & 36 \\ 8 & 26 & 112 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 9 \\ 8 & 26 & 112 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 9 \\ 0 & 10 & 40 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 9 \\ 0 & 1 & 4 \end{array}\right)$$

$\therefore \tilde{x} = (a_0, a_1)^t = (1, 4)^t$

Line of best fit : $y = 1 + 4x$

Let p_i be the point on the "line of best fit" at $x = t_i$, and let $e_i := b_i - p_i$ $(i = 1, 2, 3, 4)$. Then

$E := e_1^2 + e_2^2 + e_3^2 + e_4^2$
$= 1 + 9 + 25 + 9 = 44$

i	1	2	3	4
t_i	0	1	3	4
p_i	1	5	13	17
b_i	0	8	8	20
e_i	-1	3	-5	3

⟨Table $\boxed{\#1}$ − 1⟩

$\boxed{\#2}$ $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$

is equivalent to the following system of equation

$$\begin{cases} a_0 = 0 \\ a_0 + a_1 = 8 \\ a_0 + 3a_1 = 8 \\ a_0 + 4a_1 = 20 \end{cases}$$

which is not solvable. If we replace $(0, 8, 8, 20)^t = b^t$ by $(1, 5, 13, 17)^t = p^t$ Then the system can be solved.

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 5 \\ 1 & 3 & 13 \\ 1 & 4 & 17 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 3 & 12 \\ 0 & 4 & 16 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So the solution is $\hat{x} = (1, 4)$.

§ 4.4

$\boxed{\#4}$ (a) Let $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ Then $QQ^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq Id_{3,3}$

(b) Let $v_1 = (1, 0) \in \mathbb{R}^2$, $v_2 = (0, 0) \in \mathbb{R}^2$

Then $\langle v_1, v_2 \rangle = 0$. On the other hand $1 \cdot v_2 = 0$

So v_1, v_2 are not linearly independent.

(c) Let $v_1 = (1, 1, 1)^t/\sqrt{3}$, $v_2 = (1, 0, 0)^t$, $v_3 = (0, 1, 0)^t$

$\Rightarrow v_1, v_2, v_3$ are linearly independent.

Do the Gram-Schmidtz Orthogonalization process to get Orthonormal basis (Regard each 3×3 matrix below as a set of 3 column vectors in $\mathbb{R}^3$). as shown below.

$$\begin{pmatrix} 1/\sqrt{3} & 1 & 0 \\ 1/\sqrt{3} & 0 & 1 \\ 1/\sqrt{3} & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1/\sqrt{3} & 2/3 & 0 \\ 1/\sqrt{3} & -1/3 & 1 \\ 1/\sqrt{3} & -1/3 & 0 \end{pmatrix} \to \begin{pmatrix} 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1 \\ 1/\sqrt{3} & -1/\sqrt{6} & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/2 \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/2 \end{pmatrix} \to \begin{pmatrix} 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \end{pmatrix}$$

$\therefore$ O.N.B. containing $(1,1,1)/\sqrt{3}$: $(1,1,1)/\sqrt{3}$, $(2,-1,-1)/\sqrt{6}$ $(0,1,1)/\sqrt{2}$

$\boxed{\#18}$ (Regard each matrix as a set of column vector, Following arrows is a Gram schmidtz process).

$$\begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1/\sqrt{2} & 1/2 & 0 \\ -1/\sqrt{2} & 1/2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 0 \\ -1/\sqrt{2} & 1/\sqrt{6} & 0 \\ 0 & -2/\sqrt{6} & 1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/3 \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/3 \\ 0 & -2/\sqrt{6} & 1/3 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{12} \\ 0 & 0 & -3/\sqrt{12} \end{pmatrix}$$

Hence $\begin{cases} A = (\sqrt{2}/2, -\sqrt{2}/2, 0, 0)^t \\ B = (\sqrt{6}/6, \sqrt{6}/6, -2\sqrt{6}/6, 0)^t \\ C = (\sqrt{3}/6, \sqrt{3}/6, \sqrt{3}/6, -\sqrt{3}/2)^t \end{cases}$ is ONB for $\langle(1,1,1,1)\rangle^\perp$

§5.1

$\boxed{\#3}$ (a) False : $A = -Id_{2\times 2}$ is counterexample

$\det(I+A) = 0$, but $1+\det A = 2$

(b) True : $\det(ABC) = \det(AB)\cdot\det(C)$
$= \det(A)\cdot\det(B)\cdot\det(C)$

(c) False : $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow AB - BA = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} - \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & -b \\ a & b-d \end{bmatrix}$

$\Rightarrow \det(AB-BA) = ab \neq 0$ if $ab \neq 0$.

$\boxed{\#13}$

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \det A_1 = 1$$

$$A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -3/2 \end{pmatrix} \quad \det A_2 = 3$$

$\boxed{\#28}$ (a) True : (A not invertible $\Rightarrow \det A = 0$
$\Rightarrow \det(AB) = 0 \Rightarrow AB$ not invertible)

(b) False : $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{cases} \det A = (-1) \\ \text{(product of pivot of } A) = 1\cdot 1 = 1 \end{cases}$

(c) False : $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

$\det(A-B) = 4$

$\det A - \det B = 0$

(d) True : $\det(AB) = \det A \cdot \det B = \det B \cdot \det A = \det(BA)$

§5.2

$\boxed{\#1}$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ $\det A = 1\cdot1\cdot1 - 1\cdot2\cdot2 - 2\cdot3\cdot1$
$+ 2\cdot2\cdot3 + 3\cdot3\cdot2 - 3\cdot1\cdot3$
$= 1 - 4 - 6 + 12 + 18 - 9 = 12$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix} \qquad \det B = 1\cdot4\cdot7 - 1\cdot4\cdot6 - 2\cdot4\cdot7 + 2\cdot4\cdot5 + 3\cdot4\cdot6 - 3\cdot4\cdot5$$
$$= 28 - 24 - 56 + 40 + 72 - 60$$
$$= 0$$

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \det C = 1\cdot1\cdot0 - 1\cdot0\cdot0 - 1\cdot1\cdot0 + 1\cdot0\cdot1 + 1\cdot1\cdot0 - 1\cdot1\cdot1$$
$$= 0 - 0 - 0 + 0 + 0 - 1 = -1$$

$\therefore$ Row of A, C are independent
Row of B are not independent.

#4

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

There are only two possible way to choose nonzero from different rows and columns.

① $\begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}$ ② $\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$

① corresponds to odd permutation, and ② corresponds to even permutation (of 4 column)

So $\det(A) = -(1\cdot1\cdot1\cdot1) + (1\cdot1\cdot1\cdot1) = 0$.

$$B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

Similarly, there are only two possible way.

① $\begin{bmatrix} 1 & & & \\ & & 4 & \\ & 4 & & \\ & & & 1 \end{bmatrix}$ ② $\begin{bmatrix} & & & 2 \\ & & 4 & \\ & 4 & & \\ 2 & & & \end{bmatrix}$

① corresponds to odd permutation, ② corresponds to even.

Hence $\det(B) = -(1\cdot4\cdot4\cdot1) + (2\cdot4\cdot4\cdot2) = 48$.

#5 - In general, if we place n zeros in the first row of n by n matrix A, it is directly checked that $\det(A)=0$. We prove that n is the smallest such number. i.e, given an arbitrary r $(r<n)$ position in matrix, we can find matrix A whose entry are 0 at assigned r position, with $\det A \neq 0$.

Induction on n:

For $n=1,2$ it is straightforward.

Assume this is true for $n \leq k-1$. Now let $A \in \mathrm{Mat}_{k,k}(\mathbb{R})$ with $r<k$ position for 0. Choose one of the r position. Since $r<k$, we can find an entry of A which is in the same column of the chosen position, but is not any of r position. Assign value 1 to this entry. By removing the column and row that this entry belongs to, we get $(k-1)$ by $(k-1)$ matrix $\tilde{A}$, with at most $r-1$ position. Since $r-1<k-1$, by induction hypothesis, we can fill the entry of this submatrix such that the entries are 0 at assigned position, with $\det(\tilde{A}) \neq 0$. Now fill the removed column and row from the original matrix by 0 (except on the intersection). Then $\det(A)$ is either $\det(\tilde{A})$ or $-\det(\tilde{A})$ by cofactor formula, which is nonzero. ▨ (Induction).

- If we place n zero on every off diagonal, (n^2-n zeros) we still have A with $\det(A) \neq 0$ for example $A = \mathrm{Id}_{n,n}$. To see that n^2-n is the best number, note that if we have more than n^2-n zero in matrix, we have at least one zero column. (Hence $\det = 0$).