HW 8

Complete solution

Math 222

Fall 2016

§5.2

#11 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow$ Cofactor matrix of $A\ (=C) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$.

$AC = \begin{bmatrix} ad-b^2 & -ac+ab \\ cd-bd & -c^2+ad \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ Cofactor matrix of $B\ (=D) = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$.

$\det B = 7 \cdot (12-15) = -21$

#14 If n is odd, every term appearing in the cofactor formula for C_n is zero. So $\det(C_n)=0$ for (n: odd).
If n is even, the only nonzero term appearing in the cofactor formula for C_n is obtained by the corresponding permutation $(1\ 2)(3\ 4)\cdots(n-1\ n)$.
Since the signature of this permutation is $[n/2]$, we conclude

$$\det(C_n) = \begin{cases} 1 & (\text{for } n=4,8,\cdots) \\ -1 & (\text{for } n=2,6,\cdots) \end{cases}$$

$$C_n = \begin{pmatrix} 0 & 1 & 1 & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & 1 & \\ & & 1 & 0 & 1 & \\ & & & & \ddots & 1 \\ & & & & 1 & 0 \quad 1 \\ & & & & & 1 \quad 0 \end{pmatrix}$$

"Nonzero term in cofactor formula for C_n is obtained by multiplying numbers in the circle ◯"

#23 (a) Given 4×4 matrix $X = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ with block matrix (2×2) A, B, D

Let us write (i,j)th entry of X by X_{ij} ($1 \le i,j \le 4$). Then

$$\det(X) = \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) \underbrace{X_{1\sigma(1)} \cdots X_{4\sigma(4)}}_{(*)}$$

Note that $(*)$ vanishes unless $\{\sigma(1),\sigma(2)\} = \{1,2\}$ and $\{\sigma(3),\sigma(4)\} = \{3,4\}$. Ignoring those zero terms we can rewrite it as

$$\det(X) = \sum_{\sigma \in S_2} \operatorname{sgn}(\sigma) X_{1\sigma(1)} X_{2\sigma(2)} \cdot \sum_{\sigma' \in S_2} \operatorname{sgn}(\sigma') X_{3\sigma'(3)} X_{4\sigma'(4)}$$

$$= \det(A) \cdot \det(D)$$

(σ' is a permutation of 2 letter $\{3,4\}$)

(b) $\det\left(\begin{array}{cc|c} \multicolumn{2}{c|}{Id_{2,2}} & Id_{2,2} \\ \hline 1 & 0 & \\ 0 & 0 & Id_{2,2} \end{array}\right) = 0 \neq \det(Id_{2,2})\det(Id_{2,2}) - \det(Id_{2,2})\cdot\det\begin{pmatrix}1&0\\0&0\end{pmatrix}$

(c). It suffices to check that there exists $A, B \in Mat_{2\times 2}(\mathbb{R})$ such that

$$\det\left(\begin{array}{c|c} B & A \\ \hline A^{-1} & B^{-1} \end{array}\right) \neq 0.$$

$A = \begin{pmatrix}1&1\\0&1\end{pmatrix}, \quad B = \begin{pmatrix}0&1\\1&0\end{pmatrix}$

$$\Rightarrow \det\left(\begin{array}{cc|cc} 0&1&1&1\\ 1&0&0&1\\ \hline 1&-1&0&1\\ 0&1&1&0 \end{array}\right) = \det\left(\begin{array}{cc|cc} 0&0&0&1\\ 1&0&0&1\\ \hline 1&-1&0&1\\ 0&1&1&0 \end{array}\right) = \det\begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 1&-1&0&0\\ 0&1&1&0 \end{pmatrix}$$

$$= \det\begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&-1&0&0\\ 0&0&1&0 \end{pmatrix} \neq 0.$$

#28

$1\cdot 5\cdot 9 = 45 \qquad 3\cdot 5\cdot 7 = 105$

$2\cdot 6\cdot 7 = 84 \qquad 1\cdot 6\cdot 8 = 48$

$3\cdot 4\cdot 8 = 96 \qquad 2\cdot 4\cdot 9 = 72.$

$\Rightarrow D = -3 - 21 + 24 = 0$

This matrix is not invertible because its rows are linearly dependent. (which can also be seen from the determinant)

§5.3

#1

(a) $\begin{cases} 2x_1 + 5x_2 = 1 \\ x_1 + 4x_2 = 2 \end{cases}$

$x_1 = \frac{1}{3}\cdot\det\begin{pmatrix}1&5\\2&4\end{pmatrix} = \frac{1}{3}\cdot(-6) = -2$

$x_2 = \frac{1}{3}\det\begin{pmatrix}2&1\\1&2\end{pmatrix} = 1$

(b). $\begin{cases} 2x_1 + x_2 = 1 \\ x_1 + 2x_2 + x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases}$

$$\det\begin{bmatrix} 2&1& \\ 1&2&1 \\ &1&2 \end{bmatrix} = 2\cdot 3 - 1\cdot 2 = 4$$

$x_1 = \frac{1}{4}\cdot\det\begin{bmatrix}1&1&0\\0&2&1\\0&1&2\end{bmatrix} = 3/4 \quad x_2 = \frac{1}{4}\cdot\det\begin{bmatrix}2&1&0\\1&0&1\\0&0&2\end{bmatrix} = -1/2$

$$x_3 = \frac{1}{4} \cdot \det \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = 1/4$$

$\boxed{\#6}$ (a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{bmatrix}$ $\quad \det A = 3$

$$\Rightarrow \mathrm{Adj}(A) = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & -7 \\ 0 & 0 & 3 \end{bmatrix}^t \quad \Rightarrow A^{-1} = \frac{1}{3} \cdot \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & -7 \\ 0 & 0 & 3 \end{bmatrix}^t$$

(b). $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ $\quad \det A = 6 + (-2) = 4$

$$\Rightarrow \mathrm{Adj}(A) = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \Rightarrow A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$\boxed{\#7}$ Every cofactor of A are 0

$\Rightarrow \det(A) = \sum (\text{sign}) \cdot \underbrace{(\text{cofactor})}_{(=0)} = 0$

$\Rightarrow A$ is not invertible.

For the second, statement, Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ none of whose cofactor is 0, but A is not invertible.

$\boxed{\#19}$ Areas can be computed by determinant. Since corresponding matrices are transpose to each other, they have same area.

$$\det \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} = \det \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = 4$$

$\boxed{\#20}$ Rows are orthogonal, each has length $\sqrt{4}$.

$\Rightarrow |H| = (\sqrt{4})^4 = 16$

(This makes sense because there exists orthogonal matrix that sends orthognal basis to basis which are parallel to the standard basis e_1, e_2, e_3, e_4 and every orthogonal matrix has $\det = \pm 1$.
Or one can directly compute the determinant of H)