Math 222

HW 9

Complete Solution

§ 6.1

#4

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

$$\det(A-\lambda I) = \det\begin{bmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{bmatrix} = \lambda^2+\lambda-6 = (\lambda+3)(\lambda-2)$$

$\lambda_1 = 2$

$$\left(\begin{array}{cc|c} -3 & 3 & 0 \\ 2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad V_1 = (1,-1)^t$$

$\lambda_2 = -3$

$$\left(\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad V_2 = (-3,2)^t$$

$$A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$$

$$\det(A^2-\lambda I) = \det\begin{bmatrix} 7-\lambda & -3 \\ -2 & 6-\lambda \end{bmatrix} = \lambda^2-13\lambda+36 = (\lambda-4)(\lambda-9)$$

$\lambda_1 = 4$

$$\left(\begin{array}{cc|c} 3 & -3 & 0 \\ -2 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad V_1 = (1,-1)^t$$

$\lambda_2 = 9$

$$\left(\begin{array}{cc|c} -2 & -3 & 0 \\ -2 & -3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad V_2 = (-3,\ 2)^t$$

$\Rightarrow A^2$ has the same eigenvector as A.

If A has eigenvalue λ_1, λ_2 then A^2 has eigenvalue λ_1^2, λ_2^2.

$\lambda_1^2+\lambda_2^2 = 13$ because $\lambda_1^2+\lambda_2^2 = tr(A^2)$

#12

$$P = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0.4 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(P-\lambda I) = \begin{bmatrix} 0.2-\lambda & 0.4 & 0 \\ 0.4 & 0.8-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2\cdot\lambda$$

$$\lambda_1 = 1 \quad \left[\begin{array}{ccc|c} -0.8 & 0.4 & 0 & 0 \\ 0.4 & -0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore V = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 0 \quad \left[\begin{array}{ccc|c} 0.2 & 0.4 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore \; v = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$\Rightarrow$ eigenvectors are $(1,2,0)^t$, $(0,0,1)^t$, $(2,-1,0)^t$

$(1,2,0)^t + (0,0,1)^t = (1,2,1)^t$ is also eigenvector corresponding to eigenvalue $= 1$ with nonzero component.

#14. $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\det(Q - \lambda I) = \det \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} = \lambda^2 - 2\cos\theta \cdot \lambda + 1$$

$\Rightarrow$ solution to characteristic equation $= \cos\theta \pm i\sin\theta$

$\lambda_1 = \cos\theta + i\sin\theta$

$$\left(\begin{array}{cc|c} -is & -s & 0 \\ s & -is & 0 \end{array}\right) \to \left(\begin{array}{cc|c} i & 1 & 0 \\ 1 & -i & 0 \end{array}\right) \to \left(\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad v = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$\lambda_2 = \cos\theta - i\sin\theta$

$$\left(\begin{array}{cc|c} is & -s & 0 \\ s & is & 0 \end{array}\right) \to \left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right) \to \left(\begin{array}{cc|c} i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad v = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$\therefore$ eigenvectors $= \left\{ \begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix} \right\}$

#25 Let $x = \sum c_i x_i$ where x_i are common eigenvector to A, B with eigenvalue λ_i.

$$Ax = \sum c_i A x_i = \sum c_i \lambda_i x_i = \sum c_i B x_i = Bx \quad \Rightarrow A = B.$$

#35 Recall cofactor formula, for $A = (a_{ij}) \in Mat_{3,3}(\mathbb{R})$

$\det(A) = \sum_{\sigma \in S_3} sgn(\sigma) \cdot a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} \cdots (*)$

For permutation matrix, only one of the $3! (= 6)$ terms appearing in the right hand side of $(*)$ is nonzero, which is either $+1$, or -1. These are all realized

$$\left(\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1. \quad \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = -1 \right)$$

$\Rightarrow$ So possible determinants $= \{1, -1\}$
$\Rightarrow$ Pivots are always 1.

Since every entry of permutation matrix is less than 1,
Candidate for trace of matrix is 0, 1, 2, 3.
$\mathrm{tr}\,P = 2$ is impossible because two 1's on the diagonal entry forces the remaining diagonal entry to be 1.
$\mathrm{tr}\,P = 0, 1, 3$ are realized

$$\left(\mathrm{tr}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 0, \quad \mathrm{tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = 1, \quad \mathrm{tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3 \right)$$

$\Rightarrow$ So possible traces $= \{0, 1, 3\}$

Every 3×3 permutation matrices has either order 2 or 3 (meaning $P^2 = I$ or $P^3 = I$) So if λ is eigenvalue of P then $\lambda^3 = 1$ or $\lambda^2 = 1$. Candidate for λ are $\pm 1, e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}$ and these are all realized

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ has eigenvalue } \{1, -1\}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ has eigenvalue } \{e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}, 1\}$$

$\Rightarrow$ Possible eigenvalues $= \{1, -1, e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}\}$

§ 6.2

$\boxed{\#2}$

$$A \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$$

$\boxed{\#11}$ (a) True : A is invertible $\Leftrightarrow \det A \neq 0$
$\Leftrightarrow$ 0 is not eigvalue of A

(b) False

$$A = \left[\begin{array}{cc|c} 2 & 1 & \\ & 2 & \\ \hline & & 5 \end{array}\right] \quad \text{is not diagonalizable}$$

$$A-2I = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{array}\right] \Rightarrow \ker(A-2I) = \langle (1,0,0) \rangle$$

is one dimensional

So there does not exists eigenspace decomposition.

(i.e. $\mathbb{R}^3 \neq \ker(A-3I) \oplus \ker(A-2I)$)

(c) False

$$A = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 5 \end{bmatrix} \quad \text{is diagonalizable. (already diagonal)}$$

#16 $A_1 = \begin{bmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{bmatrix}$

$$\det(A_1 - \lambda I) = \begin{bmatrix} 0.6-\lambda & 0.9 \\ 0.4 & 0.1-\lambda \end{bmatrix} = \lambda^2 - 0.7\lambda - 0.3 = (\lambda-1)(\lambda+0.3)$$

$$\lambda = 1 \quad \left[\begin{array}{cc|c} -0.4 & 0.9 & 0 \\ 0.4 & -0.9 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -4 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad v = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

$$\lambda = -0.3 \quad \left[\begin{array}{cc|c} 0.9 & 0.9 & 0 \\ 0.4 & 0.4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A \cdot \begin{pmatrix} 9 & 1 \\ 4 & -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 9 & 1 \\ 4 & -1 \end{pmatrix}}_{=S} \underbrace{\begin{pmatrix} 1 & \\ & -0.3 \end{pmatrix}}_{=\Lambda}, \quad A = S \Lambda S^{-1}$$

$$\Lambda^k = \begin{pmatrix} 1 & 0 \\ 0 & (-0.3)^k \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{as } k \to \infty$$

So $S \Lambda^k S^{-1} \longrightarrow \begin{pmatrix} 9 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -4 & 9 \end{pmatrix} \Big/ (-13)$ as $k \to \infty$

$$= \begin{pmatrix} 9 & 0 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -4 & 9 \end{pmatrix} \Big/ (-13)$$

$$= \begin{pmatrix} 9 & 9 \\ 4 & 4 \end{pmatrix} \Big/ (13)$$

In the column you see the eigenvector corresponding to the eigenvalue $= 1$.

#18 $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1)$

$\lambda = 1 \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 3 \quad \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$A \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$\Rightarrow A = S\Lambda S^{-1} \qquad S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$\Rightarrow A^k = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \Big/ (-2)$

$= \begin{pmatrix} 1 & 3^k \\ 1 & -3^k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Big/ 2$

$= \begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix} \Big/ 2$

#31 (a) $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

$(A-aI)(A-dI) = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow A^2 - A - I = 0$

So Cayley Hamilton theorem is true for (a), (b).

#32 If A is diagonalizable with eigenvalue $\lambda_1, \cdots, \lambda_n$

$\Rightarrow (A-\lambda_1 I) \cdots (A-\lambda_n I)$

$= S(\Lambda - \lambda_1 I) \cdots (\Lambda - \lambda_n I) \cdot S^{-1} \qquad \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$

$\cdots\cdots (*)$

Note that $(\Lambda - \lambda_i I) = \begin{pmatrix} * & & & & 0 \\ & \ddots & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & * \end{pmatrix} \leftarrow$ i-th row

a diagonal matrix with i-th entry 0 (on diagonal)

So $(\Lambda-\lambda_1 I)\cdots(\Lambda-\lambda_n I) = 0$ and hence $(*) = 0$.