Spectral Geometry Spring 2016

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Lecture 1: Laplacian type operators

Let (M^n, g) be a complete oriented Riemannian manifold. By the Hopf-Rinow Theorem this means that the exponential map is defined on the whole fibre to any point, and means that geodesics do not terminate.

Definition 1. Let $E \to M$ be a smooth vector bundle over M. A connection on E is an $\mathbf{R}$-linear map

$$\nabla : C^\infty(M, E) \to C^\infty(M, E \otimes T^*M)$$

that satisfies the Leibniz rule

$$\nabla(f\sigma) = \sigma \otimes df + f\nabla\sigma$$

for $f \in C^\infty(M)$, $\sigma \in C^\infty(M, E)$.

Remark 2. This notation also captures the exterior derivative on functions, viewing smooth functions as sections of the trivial line bundle.

We usually view ∇ as a map $C^\infty(M, E) \to C^\infty(M, E \otimes T^*M)$. For $\sigma \in C^\infty(M, E)$ and X a vector field , we write $\nabla_X\sigma \in C^\infty(M, E)$ for the evaluation of the T^*M-valued one form $\nabla\sigma$ at X. Also note for fixed vector field X, ∇_X is a linear endomorphism of $C^\infty(M, E)$. Recall the fundamental theorem of Riemannian geometry.

Theorem 3. *There is a unique connection*

$$\nabla : TM \to TM \otimes T^*M$$

on the tangent bundle to M that is metric:

$$d(g(\alpha, \beta)) = g(\nabla\alpha, \beta) + g(\alpha, \nabla\beta)$$

and torsion free:

$$\nabla_X Y - \nabla_Y X - [X, Y] \equiv 0.$$

This is called the Levi-Civita connection.

One has a chain complex

$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(M)$$

where $\Omega^i(M)$ are the smooth i-forms. We will also work with the subcomplex of compactly supported forms:

$$C_c^\infty(M) \xrightarrow{d} \Omega_c^1(M) \xrightarrow{d} \Omega_c^2(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega_c^n(M).$$

In the case of compact M there is of course no distinction drawn.

We are now going to define another operator δ that maps in the opposite direction to d. The orientation of M determines a top dimensional volume form. In oriented local coordinates $x_1, \dots, x_n$ this is given by

$$\mathrm{Vol} = \sqrt{|\det g|} dx_1 \dots dx_n.$$

We now introduce the Hodge star operator $\star : \Omega^i(M) \to \Omega^{n-i}(M)$. It is the unique operator so that for $\alpha, \beta \in \Omega^i(M)$

$$\alpha \wedge \star\beta = g(\alpha, \beta)\mathrm{Vol}$$

where $g(\bullet, \bullet)$ is the local bilinear form on $\Omega^i(M)$ induced by g. Note that this is a local definition. It is easy to check on Ω^i that $\star\star = (-1)^{i(n-i)}$. Now note that for $\alpha \in \Omega_c^i(M), \beta \in \Omega_c^{i+1}(M)$,

$$\int_M g(\alpha, \star d \star \beta)\mathrm{Vol} = \int_M \alpha \wedge \star\star d \star \beta = (-1)^{i(n-i)} \int_M \alpha \wedge d \star \beta.$$

On the other hand

$$d(\alpha \wedge \star\beta) = d\alpha \wedge \star\beta + (-1)^{(n-i-1)} \alpha \wedge d \star \beta.$$

By Stoke's theorem, the integral of the above is zero, so

$$\int_M g(\alpha, \star d \star \beta)\mathrm{Vol} = (-1)^{i(n-i)+(n-i)} \int d\alpha \wedge \star\beta = (-1)^{(i+1)(n-i)} \int g(d\alpha, \beta)\mathrm{Vol}.$$

Therefore, with respect to the bilinear form

$$\langle \alpha_1, \alpha_2 \rangle \equiv \int_M g(\alpha_1, \alpha_2)\mathrm{Vol},$$

we have for $\alpha \in \Omega_c^i(M), \beta \in \Omega_c^{i+1}(M)$

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$$

where $\delta = (-1)^{(i+1)(n)} \star d\star$ on Ω^{i+1} ($i(i+1)$ is always even).

Definition 4. The *Laplace-de Rham operator* is defined on each $\Omega_c^i(M)$ as

$$\Delta = \delta d + d\delta.$$

When $i = 0$ this is called the Laplace-Beltrami operator (or just Laplacian on M).

There is another way that a Laplacian type operator can arise on a vector bundle E with connection ∇. The tensor product of any two connections is a well defined connection, hence the Levi-Civita connection along with ∇ gives a connection on $E \otimes T^*M$. Then for any $\sigma \in C^\infty(M, E)$ one has $\nabla^2\sigma$ an E valued two tensor. The metric defines a trace on these two forms that has values in $C^\infty(M, E)$.

Definition 5. The *connection (or trace) Laplacian* is given by

$$\Delta = -\mathrm{tr}\nabla^2.$$

The symbol of a differential operator.

We follow Lecture notes of Pierre Albin [1] here in our development of the symbolic calculus.

Suppose that D is a differential operator on a smooth vector bundle $E \to M$. By this we mean that there is some connection ∇ on E and D is generated by sections of $\mathrm{End}(E)$ and some ∇_{X_i} where X_i are vector fields. The differential operators of order k on E are defined to be the $C^\infty(M, \mathrm{End}(E))$ module

$$\mathrm{Diff}^k(M, E) \equiv \langle \nabla_{X_1}\nabla_{X_2}\dots\nabla_{X_j} : j \le k\rangle_{C^\infty(M,\mathrm{End}(E))}.$$

There is a *principal symbol map* σ_k which maps

$$\sigma_k : \mathrm{Diff}^k(M, E) \to C^\infty(T^*M, \pi^*\mathrm{End}(E)).$$

We define this first for an operator

$$D = a\nabla_{X_1}\nabla_{X_2}\dots\nabla_{X_j}$$

where $j \le k$, a is a local section of $\mathrm{End}(E)$ and X_i are locally defined vector fields. We define the principal symbol of D at $(\zeta, \xi) \in T^*M$ to be the element of $\mathrm{End}(E)_\zeta$

$$\begin{aligned} \sigma_k(D)(\zeta, \xi) \quad &\equiv a(\zeta)\xi(X_1)\dots\xi(X_k) \quad \text{if } j = k \\ &\quad 0 \quad\quad\quad\quad\quad\quad\quad \text{else.} \end{aligned}$$

We extend this definition linearly over $\mathbf{R}$. Note that changing the connection changes the value of ∇_{X_i} by a scalar function, so the definition of σ_k is independent of the choice of connection.

References

[1] Pierre Albin - Analysis on non-compact manifolds, Lecture notes