Spectral Geometry Spring 2016

Michael Magee

Lecture 2: Laplacian type operators II.

We begin with a concrete description of the scalar laplacian. We will use the musical isomorphisms

$$\flat : TM \to T^*M, \quad \sharp : T^*M \to TM$$

induced by g. Moreover these induce $\flat : \chi(M) \to \Omega^1(M)$, $\sharp : \Omega^1(M) \to \chi(M)$, where $\chi(M)$ are the smooth vector fields on M. Then

$$\delta\flat = \mathrm{div},$$

the divergence, and

$$\sharp d = \mathrm{grad},$$

the gradient. On one forms, $\delta = (-1)^{n(1+1)} \star d \star = \star d \star$. Therefore the scalar laplacian is given by $\Delta = \star d \star d = \mathrm{div.grad}$.

Let us calculate this operator in local oriented coordinates $x_1, x_2, \ldots, x_n$. We calculate for smooth function f that

$$\star(f dx_k) = \sum_j (-1)^{j-1} g^{jk} \sqrt{|g|} f dx_1 \wedge dx_2 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n,$$

where $\widehat{\bullet}$ denotes omission. Then

$$\begin{aligned} d \star (f dx_k) &= \sum_j (-1)^{j-1} \sum_i \frac{\partial}{\partial x_i} (\sqrt{|g|} g^{jk} f) dx_i \wedge dx_1 \wedge dx_2 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n \\ &= \sum_j \frac{\partial}{\partial x_j} (\sqrt{|g|} g^{jk} f) dx_1 \wedge dx_2 \wedge \ldots \wedge dx_j \wedge \ldots \wedge dx_n. \end{aligned}$$

Then

$$\star d \star (f dx_k) = \frac{1}{\sqrt{|g|}} \sum_j \frac{\partial}{\partial x_j} (\sqrt{|g|} g^{jk} f).$$

The Laplacian on scalars is therefore given by

$$\Delta f = \star d \star df = \star d \star \sum_k \frac{\partial f}{\partial x_k} dx_k = \frac{1}{\sqrt{|g|}} \sum_{j,k} \frac{\partial}{\partial x_j} (\sqrt{|g|} g^{jk} \frac{\partial f}{\partial x_k}).$$

This is normally written as

$$\Delta = \frac{1}{\sqrt{|g|}} \partial_j (\sqrt{|g|} g^{jk} \partial_k)$$

using summation convection and $\partial_k = \partial/\partial x_k$.

Example 1. Let $\mathbb{H}^{n+1}$ be the upper half space

$$\mathbb{H}^{n+1} = \{(\vec{x}, y) \in \mathbf{R}^n \times \mathbf{R}_{>0}\}$$

with the metric

$$\frac{d\vec{x}^2 + dy^2}{y^2},$$

in other words

$$(g_{ij}) = y^{-2} \begin{pmatrix} I_n & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $|g| = y^{-2(n+1)}$ and $g^{ij} = y^2 \delta^{ij}$ where δ is the Kronecker delta symbol. This gives

$$\Delta_{\mathbb{H}^{n+1}} = y^{n+1} \left(\sum_{i=1}^{n+1} \partial_i y^{-(n+1)} y^2 \partial_i \right) = y^{n+1} \left(\sum_{i=1}^{n+1} \partial_i y^{1-n} \partial_i \right).$$

In the special case that $n = 1$, so the case of the upper half plane, this simplifies further to

$$\Delta_{\mathbb{H}^2} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Now we give some examples of interesting connections giving rise to other Laplacian type operators.

Example 2. Let $\tilde{M}$ be the universal cover of M. Given any finite dimensional representation

$$\rho : \Gamma = \pi_1(x_0, M) \to V$$

there exists a vector bundle over M whose total space is given by the quotient

$$E_\rho = \mathrm{diag}(\Gamma) \backslash \tilde{M} \times V$$

and whose fibration is the natural mapping to $\Gamma \backslash \tilde{M} = M$. Sections of this bundle can be identified with V-valued functions on $\tilde{M}$ with invariance property

$$f(\gamma x) = \rho(\gamma) f(x)$$

for all $x \in \tilde{M}$, $\gamma \in \Gamma$. So a connection on E takes Γ-equivariant V-valued functions on $\tilde{M}$ to V-valued one forms on $\tilde{M}$ with a suitable equivariance property. Such a connection is given by the exterior derivative acting on V-valued functions. A connection obtained in this way is *flat*: it squares to 0 (since $d^2 = 0$). Caution: this square is not the same as the one in the definition of the trace Laplacian, it involves the extension of ∇ along the lines of the exterior derivative.

If moreover the representation above preserved a positive definite quadratic form then this form induces a metric on E_ρ, and the connection obtained is compatible with said metric.

Example 3. Let Γ be the fundamental group of a complete Riemannian manifold, viewed as deck transformations of its universal cover. If $\phi : \Gamma \to G$ is any surjective homomorphism from Γ to a finite group G then one obtains a representation of Γ on the finite dimensional vector space spanned by the elements of G over $\mathbf{R}$. We denote this space $\mathbf{R}^G$. Write Φ for this representation of Γ.

On the other hand, the kernel of ϕ is a normal subgroup of Γ and by the theory of covering spaces gives rise to a finite cover

$$M' \to M$$

with deck transformation group G. The sections of E_Φ can be identified with smooth functions on M'. The connection we defined in Example 2 corresponds to the exterior deriviative (acting on functions) on M'. The connection laplacian is the same as the scalar Laplacian on the covering space.

Example 4. Natural examples of flat connections arise in other ways. For example, if Γ is the fundamental group of a hyperbolic 3- manifold M then it can be viewed as a discrete subgroup of $\mathrm{SL}_2(\mathbf{C})$. We can get representations of Γ from restricting those of $\mathrm{SL}_2(\mathbf{C})$, for example one can take the kth symmetric power of the standard representation of $\mathrm{SL}_2(\mathbf{C})$ on $\mathbf{C}^2$, usally denoted $\mathrm{Sym}^k(\mathbf{C}^2)$. This gives a family of vector bundles on M with flat connections.

Example 5. On one forms, we now have two candidates for the Laplacian: the Laplace-de Rham operator and the connection Laplacian for the Levi-Civita connection. We will see that these have the same principal symbol and as such differ by a lower order differential operator. In fact, this is a zeroth order operator depending only on the Riemann curvature tensor. This is known as a *Weitzenböck identity*.