Spectral Geometry Spring 2016

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Lecture 3: Elliptic regularity and densely defined operators.

Examples of symbols.

Around any point on M there exist *normal coordinates* $x_1, \ldots, x_n$ where the metric $g_{ij} = \delta_{ij}$ and the Christoffel symbols for the Levi-Civita connection vanish. The Laplace-Beltrami operator is given in local coordinates by

$$-\frac{1}{\sqrt{|g|}}\partial_i \left(\sqrt{|g|}g^{ij}\partial_j f\right) = \sum_i -\partial_i^2 f$$

therefore the principal symbol at (ζ, ξ) should be given by $\sigma_2(\Delta) = \sum_i -\xi(\partial_i)^2 = -g_\zeta(\xi, \xi)$.

As for the connection Laplacian, in a local orthonormal frame $\{e_i\}$ for E this can be calculated as

$$-\mathrm{tr}\nabla^2 = -\sum_i (\nabla_{\partial_i})^2 + \nabla_{(\nabla_{\partial_i}\partial_i)}.$$

and as such, the principal symbol is

$$\sigma_2(\Delta)(\zeta, \xi) = -\sum \xi(\partial_i)^2 = -g_\zeta(\xi, \xi)$$

where g_ζ is the fibre metric.

Definition 1. A differential operator is called elliptic if its principal symbol is invertible.

In the example above, the Laplace-Beltrami operator is elliptic since the principal symbol is given by the scalar function $(\zeta, \xi) \mapsto -g_\zeta(\xi, \xi)$. This is non zero at every point, so invertible (as a multiplication operator). The Laplace de-Rham operators are also elliptic.

For the remainder of this lecture we restrict to the Laplace-Beltrami operator. Whatever we say generalizes to a connection Laplacian or the Laplace-de Rham operators.

We are going to use a black box that follows from the theory of pseudodifferential operators. This requires the concept of a smoothing operator

Definition 2. A *smoothing operator* is an integral operator K obtained from a smooth kernel $k \in C^\infty(M \times M)$ by using the formula

$$K[f](x) = \int k(x, y)f(y)\mathrm{Vol}_y.$$

If M is compact a smoothing operator is compact on $L^2(M)$ with image in $C^\infty(M)$. For more general complete M one can say that a smoothing operator maps $L^2_c(M)$, the compactly supported L^2 functions, to $C^\infty(M)$.

Proposition 3. *If $P \in \mathrm{Diff}^k(M,E)$ is an elliptic differential operator then there exist* ***parametrices*** *Q, Q' such that*

$$\begin{aligned} QP - \mathrm{Id} &= R, \\ PQ' - \mathrm{Id} &= R' \end{aligned} \tag{1}$$

where R and R' are ***smoothing operators*** *and Q, Q' are pseudodifferential operators*[1] *of order $-k$.*

We also introduce the following *Sobolev space:* let

$$H^2(M) = \{\, u \in L^2(M) \ : \ \|\nabla u\| \in L^2(M)\,,\ \Delta u \in L^2(M)\,\}.$$

Proposition 4 (Elliptic regularity for eigenfunctions of the Laplacian)**.** *If $u \in H^2(M)$ with $\Delta u = \lambda u$ for some $\lambda \in \mathbf{R}$ then u is smooth.*

Proof. The operator $\Delta - \lambda$ is elliptic (adding λ does not change the principal symbol of the operator). Let Q be a left parametrix for $P = \Delta - \lambda$ as in 1. For any point $x \in M$ let χ be a smooth compactly supported bump function that is $\equiv 1$ on a neighborhood U of x. Then

$$QPu \equiv 0$$

on U, using the property of the pseudodifferential operator Q that it acts locally (so preserves $Pu \equiv 0$ on U). Now using the parametrix equation we get

$$\chi u + Ru \equiv 0.$$

Since Ru is smooth, u is smooth near x. As x was arbitrary, u is smooth on M. □

Theory of unbounded operators.

We are following here Reed and Simon [2, Chapter VIII]. We write (L, V) for a pair consisting of linear subspace V of a Hilbert space H and a linear map $L : V \to H$.

Definition 5. A pair (L, V) is densely defined on H if V is dense in H.

For example, we could take the pair $(\Delta, C_c^\infty(M))$ as a densely defined operator on the Hilbert space of square integrable functions $L^2(M, \mathrm{Vol})$. Indeed this is the main example we are interested in.

Definition 6. If (L, V) is a densely defined operator on H, the graph of L is the set

$$\Gamma(L, V) \equiv \{(v, Lv) : v \in V\} \subset V \times H.$$

The operator (L, V) is called closed if $\Gamma(L, V)$ is closed in the product topology on $H \times H$.

Definition 7. We say that (L', V') is an extension of (L, V) if $V \subset V'$ and $L'|_V \equiv L$.

Definition 8. We say a pair (L, V) is closable if it has a closed extension. Every closable operator has a 'smallest' closed extension called the closure of the operator.

[1] It is not really necessary for our purposes to precisely define this. A good reference is Pierre Albin's notes [1].

Definition 9. For (L, V) densely defined on H let V^* be the set of $\phi \in H$ such that there is an η so that

$$\langle L\psi, \phi\rangle = \langle \psi, \eta\rangle$$

for all $\psi \in V$. Then define $L^* : V^* \to H$ by $L^*(\phi) = \eta$. Since V is assumed dense, η is uniquely determined. The pair (L^*, V^*) is called the adjoint to (L, V). Warning: here V^* might not be dense.

References

[1] Pierre Albin - Analysis on non-compact manifolds, Lecture notes.

[2] Michael Reed, Barry Simon - Methods of Modern Mathematical Physics I: Functional Analysis.