Spectral Geometry Spring 2016

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Lecture 4: The spectral theorem.

Recall we consider pairs (L, V) where V is a subspace of a Hilbert space H and $L : V \to H$ is a linear operator.

Definition 1. We say (L, V) is closable if it has a closed extension. Every closable operator has a 'smallest' closed extension called the closure of the operator.

Definition 2. For (L, V) densely defined on H let V^* be the set of $\phi \in H$ such that there is an η so that

$$\langle L\psi, \phi\rangle = \langle \psi, \eta\rangle$$

for all $\psi \in V$. Then define $L^* : V^* \to H$ by $L^*(\phi) = \eta$. Since V is assumed dense, η is uniquely determined. The pair (L^*, V^*) is called the adjoint to (L, V).

Warning: V^* might not be dense.

Theorem 3 ([1, Theorem VIII.1])**.** *If (L, V) is densely defined on H then*

1. *L^* is closed.*
2. *L is closable iff V^* is dense, in which case the closure of (L, V) is (L^{**}, V^{**}).*
3. *If L is closable with closure $(\bar{L}, \bar{D})$ then the adjoint of $(\bar{L}, \bar{D})$ is the pair (L^*, V^*).*

Definition 4. A densely defined operator (L, V) is symmetric iff

$$\langle L\phi, \psi\rangle = \langle \phi, L\psi\rangle$$

for all ϕ, ψ in V.

Lemma 5. *Let Δ denote the Laplace-de Rham operator. $(\Delta, \Omega^i_c(M))$ is symmetric.*

Proof. It is enough to check for α, β in $\Omega^i_c(M)$ that

$$\langle \Delta\alpha, \beta\rangle = \langle (d\delta + \delta d)\alpha, \beta\rangle = \langle \delta\alpha, \delta\beta\rangle + \langle d\alpha, d\beta\rangle = \langle \alpha, (d\delta + \delta d)\beta\rangle = \langle \alpha, \Delta\beta\rangle.$$

□

Fact 6. *A symmetric operator is always closable since (by symmetry) every element in V is in V^*: we can just set $\eta = T\phi$. Therefore V^* is dense so Theorem 3 applies.*

Definition 7. (L, V) is *self-adjoint if L is symmetric and $V^* = V$.*

Definition 8. (L, V) is *essentially self-adjoint* if its closure is self-adjoint.

We note here that an essentially self-adjoint operator has a unique self adjoint extension. (Any S a self adjoint extension of T extends T^{**}. Then $S = S^*$ is extended by $(T^{**})^* = T^{**}$. So then $S = T^{**}$.)

Theorem 9. *The Laplace de-Rham operator* $(\Delta, \Omega_c^i(M))$ *is essentially self adjoint.*

We'll follow the proof of Strichartz from [3], using the following criterion.

Lemma 10 ([2, Theorem X.1])**.** *If* (L, V) *is closed, positive definite, symmetric and densely defined, then* $(L, V) = (L^*, V^*)$ *iff there are no eigenvectors with negative eigenvalue in* V^*.

Let $\bar{\Delta}$ be the closure of the Laplace de-Rham operator $(\Delta, \Omega_c^i(M))$ in the Hilbert space L^2. Let V be its domain, and V^* the domain of its adjoint. By Theorem 3, V^* is the domain of the adjoint to $(\Delta, \Omega_c^i(M))$, that is, those L^2 forms v for which the distribution Δv can be identified with an L^2 section, as per our definition of the adjoint. We will apply the criterion of Lemma 10 to the closed, positive definite, symmetric and densely defined operator $\bar{\Delta}$.

Suppose now that $(\bar{\Delta})^* v = \lambda v$ for some $\lambda < 0$. Since v is the weak solution to an elliptic eigenvalue equation, elliptic regularity tells us that in fact, v is infinitely differentiable. Let ϕ be a compactly supported test function. Direct calculation involving integration by parts gives

$$0 \geq \lambda \langle \phi^2 v, v \rangle = \langle \phi^2 dv, dv \rangle + \langle \phi^2 \delta v, \delta v \rangle + 2 \langle \phi d\phi \wedge v, \delta v \rangle - 2 \langle v, \phi d\phi \wedge \delta v \rangle.$$

Then by Cauchy-Schwarz and so on,

$$\|\phi dv\|_2^2 + \|\phi \delta v\|_2^2 \leq 2 \|d\phi\|_\infty \|uv\| (\|\phi dv\|_2 + \|\phi \delta v\|).$$

Then

$$\|\phi dv\|_2 + \|\phi \delta v\|_2 \leq 4 \|d\phi\|_\infty \|v\|_2.$$

Around every point we can find a family of compactly supported functions that are each $\equiv 1$ on a fixed neighborhood of that point and with $\|d\phi\|_\infty \to 0$. Doing this for each point then changing the point implies $dv \equiv 0$ and $\delta v \equiv 0$, hence $v = \lambda^{-1} \Delta v \equiv 0$. This shows there can be no weak eigenvector with negative eigenvalue, which completes the proof that Δ is essentially self-adjoint.

From now on, on a complete Riemann manifold, we just write Δ *for the unique self adjoint extension of the Laplace-de Rham operator from compactly supported sections to* L^2.

The spectral theorem.

Definition 11 (Projection valued measure)**.** A projection valued measure on a Hilbert space H is a function $\Omega \to P_\Omega$ from the Borel measurable sets $\mathcal{B}(\mathbf{R})$ on $\mathbf{R}$ to the bounded operators on H such that

1. For each Ω in $\mathcal{B}(\mathbf{R})$, P_Ω is an orthogonal projection.

2. $P_\emptyset = 0$, $P_{\mathbf{R}} = I$.

3. If Ω is the countable disjoint union $\Omega = \coprod_{i=1}^\infty \Omega_n$ then $P_\Omega = \lim_{N \to \infty} \sum_{i=1}^N P_i$ where the limit is in the strong operator topology.

4. $P_{\Omega_1} \cap P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.

Notice given a projection valued measure, and a Borel measurable *function* $g \in \mathbf{R}$ that will be denoted by the formula

$$P[g] = \int_{\mathbf{R}} g(\lambda) dP(\lambda).$$

This is just formalism so far - the definition is as follows. Note that for all ϕ, ψ in H one can form a Borel measure $\mu_{\phi,\psi}$ by defining

$$\mu_{\phi,\psi}(\Omega) = \langle P_\Omega \phi, \psi \rangle.$$

Then we define

$$\langle P[g]\phi, \psi \rangle \equiv \int_{\mathbf{R}} g(\lambda) d\mu_{\phi,\psi}(\lambda).$$

Knowing all these matrix coefficients defines an operator on the domain

$$D_g \equiv \{\phi : \int_{\mathbf{R}} |g(\lambda)|^2 d\mu_{\phi,\phi}(\lambda) < \infty\}.$$

If g is real valued, then $P[g]$ is self-adjoint on D_g.

Theorem 12 (Spectral Theorem for unbounded operators, [1, Theorem VIII.6])**.** *The mapping*

$$P \mapsto \int_{\mathbf{R}} \lambda dP(\lambda)$$

gives a one-to-one correspondence between projection valued measures on H and self adjoint operators on H.

References

[1] Michael Reed, Barry Simon - Methods of Modern Mathematical Physics I: Functional Analysis.

[2] Michael Reed, Barry Simon - Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness.

[3] Robert S. Strichartz. Analysis of the Laplacian on the Complete Riemannian Manifold. Journal of Functional Analysis **52**, 48-79 (1983)