Spectral Geometry Spring 2016

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Lecture 5: The resolvent.

Last time we proved that the Laplacian has a closure whose domain is the Sobolev space

$$H^2(M) = \{u \in L^2(M) \ : \ \|\nabla u\| \in L^2(M),\ \Delta u \in L^2(M)\ \}.$$

We view the Laplacian as an unbounded densely defined self-adjoint operator on L^2 with domain H^2. We also stated the spectral theorem. We'll now ask what is the nature of the projection valued measure associated to the Laplacian (its 'spectral decomposition'). Until otherwise specified, suppose M is a complete Riemannian manifold.

Definition 1. The *resolvent set* $\mathcal{R}$ of the Laplacian is the set of $\lambda \in \mathbf{C}$ such that $\lambda I - \Delta$ is a bijection of H^2 onto L^2 with a bounded inverse (with respect to L^2 norms)

$$R(\lambda) \equiv (\lambda I - \Delta)^{-1} : L^2(M) \to H^2(M).$$

The operator $R(\lambda)$ is called the *resolvent* of the Laplacian at λ. We write

$$\mathrm{spec}(\Delta) = \mathbf{C} - \mathcal{R}$$

and call it the spectrum.

Note that the projection valued measure of Δ is supported in $\mathbf{R} \geq 0$. Otherwise one can find a function ψ in $H^2(M)$ where $\langle \Delta\psi, \psi\rangle$ is negative, but since ψ can be approximated in Sobolev norm by smooth functions, this will contradict

$$\langle \Delta\psi, \psi\rangle = \int g(\nabla\psi, \nabla\psi)\mathrm{Vol} \geq 0. \tag{1}$$

Note then the spectral theorem implies that the resolvent set contains $\mathbf{C} - \mathbf{R}_{\geq 0}$ (use the Borel functional calculus).

Theorem 2 (Stone's formula [1, VII.13])**.** *Let P be the projection valued measure on $\mathbf{R}$ associated to the Laplacan by the spectral theorem. The following formula relates the resolvent to P*

$$\lim_{\epsilon\to 0}(2\pi i)^{-1}\int_a^b \left[R(\lambda + i\epsilon) - R(\lambda - i\epsilon)\right] d\lambda = \frac{1}{2}\left[P_{[a,b]} + P_{(a,b)}\right].$$

The limit is taken in the strong operator topology and the integral can be interpreted as a limit of Riemann sums in the strong operator topology.

We want to study the operator $R(-1) = -(\Delta + 1)^{-1}$. This is a bounded self adjoint operator from $L^2(M)$ to $L^2(M)$ with image in $H^2(M)$. The following will be useful.

Theorem 3 (Rellich-Kondrachov embedding theorem)**.** *On a compact M of dimension n, if $k > l$ and $k - n/p > l - n/q$ then the map*

$$W^{k,p} \to W^{l,q}$$

is compact.

Assume now M is compact. Up until now we only know that $R(-1)$ is only bounded from L^2 to L^2 and that it happens to map into H^2. We can use our parametrix from before to improve this statement. It is convenient to introduce the notation $\Delta(\lambda) = (\lambda - \Delta)$ and let Q be the left parametrix to $\Delta(1)$, so that

$$Q\Delta(-1) = I + K$$

where K is a compact smoothing remainder. Now given $f \in L^2$ consider that

$$\Delta(-1)R(-1)f = f$$

so

$$(I + K)R(-1)f = Q\Delta(-1)R(-1)f = Qf.$$

Then

$$R(-1)f = (Q - K)f.$$

Since K is a bounded map to any Sobolev space and Q is a bounded map from L^2 to H^2 we obtain the result.

Lemma 4. *If M is compact then $R(-1)$ is a bounded map from $L^2(M)$ to $H^2(M)$ with respect to the Sobolev norm.*

Using this Lemma along with Theorem 3 with $p = q = 2$ and $k = 2$, $l = 0$ gives

Proposition 5 ('The Laplacian has compact resolvent')**.** *If M is compact, the resolvent $R(-1)$ is compact from $L^2(M)$ to $L^2(M)$.*

Theorem 6 (Spectral theorem for compact normal operators)**.** *If T is a compact normal operator on a Hilbert space H then the spectrum of T (the values of $\lambda \in \mathbf{C}$ for which $T - \lambda$ is not invertible) is a countable set of nonzero eigenvalues that can accumulate only at 0. If λ is in the spectrum then $\ker(\lambda - T)$ is a finite dimensional eigenspace. These eigenspaces form an orthogonal direct sum decomposition for H.*

Applying this to $R(-1)$, we get that $(\mu - R(-1))$ is invertible outside a set of real negative μ_i accumulating only at zero. Therefore

Proposition 7. *If M is compact, the projection valued measure P associated to the Laplace-Beltrami operator is atomic and supported on the set of eigenvalues $\lambda_i = -\mu_i^{-1} - 1$. These eigenvalues $\lambda_i \to \infty$ as $i \to \infty$. The value $P_{\{\lambda_i\}}$ is a projection onto a finite dimensional eigenspace. Finally, $\lambda_1 = 0$ and the eigenspace of 0 is one dimensional if M is connected.*

Proof. Only the last statement needs checking. It follows from the energy estimate (1). □

Remark 8. If M is compact, most of Proposition 7 holds for any of the other Laplacians we have mentioned. The only (but very important) difference is that 0 may or may not be in the spectrum, and it may have a multiplicity > 1. (cf. Hodge Theorem).

References

[1] Michael Reed, Barry Simon - Methods of Modern Mathematical Physics I: Functional Analysis.