Spectral Geometry Spring 2016

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Lecture 6: Flat tori I: The spectrum.

In this lecture we discuss the Laplace-Beltrami operator on flat tori. A torus is a topological manifold homeomorphic to $(S^1)^n$ for some n. A flat torus is a torus with the structure of a Riemannian manifold (M^n, g) such that the metric is flat. The usual definition of flatness is that the curvature tensor is identically zero. This implies that at every point $p \in M$ there an neighborhood $p \in U$ and a system of coordinates $x_1, \ldots, x_n$ on U so that the vector fields ∂_i form an orthonormal frame *at every* $q \in U$.

Theorem 1 (Hopf)**.** *Any complete, flat, simply connected n-dimensional manifold is isometric to $\mathbf{R}^n$ with its Euclidean metric.*

This implies that any flat torus is isometric to a quotient of $\mathbf{R}^n$ by some group G of Euclidean isometries acting freely and properly discontinuously. We also know the deck transformation group is isomorphic to $\mathbf{Z}^n$ and the quotient is compact. We have the decomposition $\mathrm{Isom}(\mathbf{R}^n) = \mathbf{R}^n \rtimes O(n)$. We say an element is a *translation* if its projection to $O(n)$ is 1, in other words, it corresponds to adding some vector to $\mathbf{R}^n$. Say G acts by translations if every element is a translation.

The following Proposition from Thurston's book [1] implies our deck transformation group G acts by translations.

Proposition 2 ([1, Proposition 4.2.4])**.** *An abelian subgroup G of $\mathrm{Isom}(\mathbf{R}^n)$ has a unique maximal subspace on which G acts by translations. If G is discrete and cocompact then this space is $\mathbf{R}^n$.*

We can therefore identify the fundamental group of M^n with a discrete subgroup

$$\Lambda \subset \mathbf{R}^n$$

where $\Lambda \cong \mathbf{Z}^n$ and Λ acts as translations on $\mathbf{R}^n$. We call such a Λ a *lattice.* So every torus arises as a quotient

$$\Lambda \backslash \mathbf{R}^n.$$

We introduce the *dual lattice*

$$\Lambda' = \{\lambda' \in \mathbf{R}^n \ : \ \langle \lambda', \lambda \rangle \in \mathbf{Z} \, \forall \lambda \in \Lambda \rangle,$$

noting that it is indeed a subgroup of $\mathbf{R}^n$ isomorphic to $\mathbf{Z}^n$. The pairing $\langle \bullet, \bullet \rangle$ is the standard Euclidean inner product on $\mathbf{R}^n$. Let us also introduce the notation $e(\theta) = \exp(2\pi i \theta)$.

We check that each of the functions

$$f_{\lambda'}(\xi) = e(\langle \lambda', \xi \rangle) \quad \lambda' \in \Lambda'$$

is invariant under

$$\xi \mapsto \xi + \lambda$$

for all $\lambda \in \Lambda$. Hence they can be viewed as functions on $M = \Lambda\backslash\mathbf{R}^n$. Also, we can compute

$$\Delta_M f_{\lambda'}$$

by computing on $\mathbf{R}^n$. This is because the metric on M comes from the Λ-invariant metric on M and hence the Laplacian on M is obtained from the Λ-invariant Laplacian on $\mathbf{R}^n$. Therefore if $\lambda' = (\lambda'_1, \ldots, \lambda'_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$ are coordinates on $\mathbf{R}^n$ then

$$\begin{aligned} \Delta_M f_\lambda &= \sum_{j=1}^n -\partial_j^2 f_{\lambda'} = \sum_{j=1}^n -\partial_j^2 \exp(2\pi i \sum_{k=1}^n \lambda'_k \xi_k) = -\sum_{j=1}^n -(2\pi)^2 (\lambda'_j)^2 f_{\lambda'} \\ &= 4\pi^2 \|\lambda'\|^2 f_{\lambda'}. \end{aligned}$$

So every $f_{\lambda'}$ is an eigenfunction of the Laplacian on M with eigenvalue $4\pi^2\|\lambda'\|^2$. They are also orthogonal to one another. If $\|\lambda'\| \neq \|\lambda\|$ this follows from our spectral theorem for compact M. Otherwise one can check it by hand by changing coordinates to the basis given by generators of Λ.

Now we need to make sure we have found all the eigenfunctions. It is enough to show the $f_{\lambda'}$ are dense in L^2, since the set of eigenfunctions we have found can be completed to a full set of orthogonal eigenfunctions. A new eigenfunction appears, we would not be able to approximate it with the $f_{\lambda'}$ in L^2.

We'll use the complex version of the Stone-Weierstrass Theorem:

Theorem 3 (Stone-Weierstrass)**.** *If A is an complex unital $*$-subalgebra of complex valued continuous functions on a compact Hausdorff space K that separates the points of K then A is dense in the complex valued continuous functions with respect to the sup norm.*

In the current setting, being dense in the sup norm implies being dense in L^2. Note that the complex unital $*$-algebra generated by the $f_{\lambda'}$ is just the complex linear span of the $f_{\lambda'}$. This is because $\overline{f_{\lambda'}} = f_{-\lambda'}$ and $f_{\lambda'_1} f_{\lambda'_2} = f_{\lambda'_1 + \lambda'_2}$. The algebra is unital because it contains $1 = f_0$.

Now we check the $f_{\lambda'}$ separate points. By change of basis we can assume $\Lambda = \mathbf{Z}^n$. Then we can reduce to the case $n = 1$ by noting any two distinct points differ at some coordinate, without loss of generality the first, and considering $\lambda' = (1, 0, 0, \ldots, 0)$. If $f_{\lambda'}(x) = e(x_1) = e(y_1) = f_{\lambda'}(y)$ then $e((x_1 - y_1)) = 1$ so $x_1 - y_1 \equiv 0 \bmod 1$, a contradiction.

We have now proved

Theorem 4. *The eigenvalues of Δ on $L^2(\mathbf{R}/\Lambda)$ are precisely the values $4\pi^2\|\lambda'\|^2$ where $\lambda' \in \Lambda'$. The $f_{\lambda'}$ are linearly independent and span the eigenspace over $\mathbf{C}$. Therefore the dimension M_μ of the eigenspace for the eigenvalue μ is the size of the set*

$$\{\lambda' \in \Lambda' : 4\pi^2\|\lambda'\|^2 = \mu\}.$$

Example 5. For the Gaussian integers ($\Lambda = \mathbf{Z} \oplus \mathbf{Z}$) M_μ is quite an irregular quantity. Indeed it is zero for any μ such that $\mu/4\pi^2$ contains a prime factor $\equiv 3 \bmod 4$ with odd multiplicity. Otherwise it is 4 times a multiplicative function of $\mu/4\pi^2$ whose values on p^n with $p \equiv 1 \bmod 4$ are $n+1$ and whose values on p^{2m} with $p \equiv 3 \bmod 4$ are 1. Therefore it is bounded by the divisor function, so for any $\epsilon > 0$ we have

$$M_\mu \ll_\epsilon \mu^\epsilon$$

where $\ll_\epsilon$ is Vinogradov notation indicating the implied constants depend on ϵ. In detail, it means there are constants $C_1(\epsilon)$ and $C_2(\epsilon)$ such that when $\mu \geq C_1(\epsilon)$, $M_\mu \leq C_2(\epsilon)\mu^\epsilon$.

References

[1] William Thurston - Three dimensional Geometry and Topology.