Spectral Geometry Spring 2016

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Lecture 7: Flat tori II: Weyl law, theta functions, heat trace.

Weyl law

In the case of flat tori $M = \Lambda\backslash\mathbf{R}^n$ there is a proof of the Weyl law by a direct counting arguments. From our previous lecture the number of eigenvalues of Δ on M less than or equal to T, counted with multiplicities is given by

$$M(T) = |\{\mu \in \operatorname{spec}(\Delta) \,:\, \mu \leq T\}| = \left\{\lambda' \in \Lambda' \,:\, \|\lambda'\| \leq \frac{T^{1/2}}{2\pi}\right\}$$

where Λ' is the dual lattice. By an elementary argument (a good exercise), for any lattice $\Lambda' \subset \mathbf{R}^n$ the number of elements with norm $\leq R$ is

$$\frac{c_n R^n}{\operatorname{Vol}(\Lambda'\backslash\mathbf{R}^n)} + o(R^n)$$

where c_n is the volume of the unit ball in $\mathbf{R}^n$. In this case the volume of $\Lambda'\backslash\mathbf{R}^n$ is $\operatorname{Vol}(\Lambda\backslash\mathbf{R}^n)^{-1} = \operatorname{Vol}(M)^{-1}$ and so we get

$$M(T) = \frac{\operatorname{Vol}(M) c_n T^{n/2}}{(2\pi)^n} + o(T^{n/2}).$$

This is *Weyl's law* for flat tori.

The Poisson summation formula

Let us define the Fourier transform for $f \in L^1(\mathbf{R})$

$$\hat{f}(\xi) = \int_{\mathbf{R}} f(x) e(-\xi x) dx.$$

Theorem 1 (Poisson Summation - [2, Theorem 4.4])**.** *Suppose that both f, $\hat{f}$ are in $L^1(\mathbf{R})$ and have bounded variation. Then*

$$\sum_{m\in\mathbf{Z}} f(m) = \sum_{n\in\mathbf{Z}} \hat{f}(n)$$

and both series are absolutely convergent.

The Poisson Summation Formula has a generalization to lattices. Let Λ be as before. Consider that for f a smooth function of rapid decay

$$\sum_{\lambda\in\Lambda} f(\lambda) = \sum_{m_1\in\mathbf{Z}}\sum_{m_2\in\mathbf{Z}}\cdots\sum_{m_n\in\mathbf{Z}} f(m_1\lambda_1+m_2\lambda_2+\ldots+m_n\lambda_n) = \sum_{m_1\in\mathbf{Z}}\sum_{m_2\in\mathbf{Z}}\cdots\sum_{m_n\in\mathbf{Z}} f\circ J_\Lambda(m_1,m_2,\ldots,m_n)$$

where J_Λ is the invertible linear map

$$(x_1,\ldots,x_n)\mapsto m_1\lambda_1+m_2\lambda_2+\ldots+m_n\lambda_n.$$

So we can apply the Poisson summation formula n times to get the above

$$= \sum_{k_1,\ldots,k_n} \widehat{f\circ J_\Lambda}(k_1,\ldots,k_n) \tag{1}$$

where $\hat{\bullet}$ is the n dimensional Euclidean Fourier transform. Now changing variables $y = J_\Lambda x$ in the Fourier transform we get

$$\begin{aligned}
\widehat{f\circ J_\Lambda}(\xi) &= \int_{\mathbf{R}^n} e(-\langle\xi,x\rangle)f(J_\Lambda x)dx = |\det J_\Lambda|^{-1}\int_{\mathbf{R}^n} e(-\langle\xi,J_\Lambda^{-1}y\rangle)f(y)dy \\
&= |\det J_\Lambda|^{-1}\int_{\mathbf{R}^n} e(-\langle (J_\Lambda^{-1})^T\xi,y\rangle)f(y)dy \\
&= |\det J_\Lambda|^{-1}\hat{f}((J_\Lambda^{-1})^T\xi).
\end{aligned}$$

Returning to (1) we get

$$= \sum_{k_1,\ldots,k_n} \widehat{f\circ J_\Lambda}(k_1,\ldots,k_n) = |\det J_\Lambda|^{-1}\sum_{k_1,\ldots,k_n}\hat{f}((J_\Lambda^{-1})^T(k_1,\ldots,k_n)).$$

A little thought reveals the latter sum is precisely the values of $\hat{f}$ over Λ' so we obtain

$$\sum_{\lambda\in\Lambda} f(\lambda) = |\det J_\Lambda|^{-1}\sum_{\lambda'\in\Lambda'}\hat{f}(\lambda'). \tag{2}$$

Note that $|\det J_\Lambda| = \mathrm{Vol}(M)$. This is because the parallelotope spanned by the λ_i is a fundamental domain for Λ acting on $\mathbf{R}^n$. The identity (2) has a range of applications.

Theta series

For any lattice Λ with a positive definite quadratic form q one can form the theta series

$$\Theta(z) = \sum_{\lambda\in\Lambda} e(q(\lambda)z/2)$$

that is parameterized by a complex $z = x+iy$. It is uniformly absolutely convergent on compact sets in $\Re(y) > 0$, so it yields a holomorphic function of z in the upper half plane.

We can always change basis so that q is the norm form on $\mathbf{R}^n$ and $\Lambda\subset\mathbf{R}^n$. Then the above series is

$$\Theta(z) = \sum_{\lambda\in\Lambda} e(\|\lambda\|^2 z/2).$$

It is possible to calculate the Fourier transform of $f(x) = e(\|x\|^2 z/2)$. To make things simpler we assume n even and q is unimodular, which means $\Lambda = \Lambda'$ and $|\det J_\Lambda| = 1$. Then

$$\hat{f}(\xi) = \frac{1}{z^{n/2}} e(-\|\xi\|^2/2z).$$

Now using PSF

$$\Theta(z) = \frac{1}{z^{n/2}} \Theta\left(\frac{-1}{z}\right).$$

On the other hand, if q is even (valued in $2\mathbf{Z}$) then also

$$\Theta(z) = \Theta(z+1).$$

The two Möbius transformations $z \mapsto 1/z$ and $z \mapsto z+1$ generate the whole modular group

$$\mathrm{SL}_2(\mathbf{Z}).$$

Since Θ satisfies the correct cocycle condition for the generators, it is a modular form of weight $n/2$. Recall a modular form of weight k is a holomorphic function on the upper half plane $\mathbb{H}$ that is holomorphic at $i\infty$ and satisfies the cocyle conditions

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

Suppose Λ is even unimodular and consider the corresponding torus. Clearly the theta series $\Theta(z)$ determines the spectrum of the Laplacian (it is easy to extract the norms and multiplicities of $\Lambda = \Lambda'$). We say

Definition 2. Two Riemannian manifolds are isospectral if their Laplace Beltrami operators have the same spectrum with multiplicities.

The following observation is due to Milnor and I am following Conway's book [1]. A result of Witt says there are exactly two distinct unimodular even 16 dimensional lattices, namely

$$\Lambda_1 = E_8 \oplus E_8$$

and

$$\Lambda_2 = D_{16}^+,$$

where D_n^+ is the integer tuples with even sum added to $(1/2, 1/2, 1/2, \ldots, 1/2)$. By the way, $E_8 = D_8^+$. Since these (quadratic) lattices are distinct, the resulting tori cannot be isometric - any isometry of the tori would extend to an isometry of the universal covers that conjugated the lattices. On the other hand, there is exactly one modular form of weight 8 for $\mathrm{SL}_2(\mathbf{Z})$, it is given by

$$\Theta(z) = 1 + 480 \sum_n \sigma_7(n) q^{2n}.$$

Therefore these tori are isospectral, but they are not isometric. This result is due to Milnor and was the first answer to Kac's question 'Can you hear the shape of a drum'. Conway's book [1] offers more on this subject.

The heat trace

Now let Λ be any lattice and consider $f(x) = \exp(-\|x\|^2/4t)$. This is essentially the same example as before. Now the fourier transform is $\hat{f}(x) = (4\pi t)^{n/2}\exp(-4\pi^2 t\|x\|^2)$. Poisson Summation gives

$$\sum_{\lambda' \in \Lambda'} \exp(-4\pi^2 t\|\lambda'\|^2) = \frac{\mathrm{Vol}(M)}{(4\pi t)^{n/2}} \sum_{\lambda \in \Lambda} \exp\left(-\frac{\|\lambda\|^2}{t}\right). \tag{3}$$

The left hand side is called the *heat trace*: it is exactly a sum over the eigenvalues μ of the laplacian of the function $\exp(-t\mu)$ and is therefore the trace of the *heat operator*

$$\exp(-t\Delta).$$

This is a version of the trace formula on the torus : the left hand side is an analytic sum over eigenvalues and the right a geometric sum over the fundamental group of a function of the length of the unique closed geodesics in the homotopy class.

One can also deduce Weyl's law from (3) by using some Tauberian Theorem. This is an approach that extends to Riemann surfaces with noncommutative fundamental groups (e.g. compact with genus ≥ 2).

References

[1] J.H. Conway. The sensual (quadratic) form.

[2] H. Iwaniec, E. Kowalski - Analytic Number Theory.