Functional Analysis (325b/525b) Midterm exam

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There are 3 questions. You have 1 hour and 15 minutes.

Question 1.

a. State and prove the Arzelà-Ascoli Theorem.
b. Prove the norm

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$$

makes the space of once continuously differentiable real valued functions $C^1[0,1]$ into a Banach space (you may assume C^0 is a Banach space). Show any $\| \bullet \|_{C^1}$-bounded set of functions in $C^1[0,1]$ has a subsequence that converges in $\| \bullet \|_\infty$.

Question 2.

a. Let X be a normed vector space. Describe the natural map from X to X^{**} and prove it is an injection that is an isometry onto its image.

b. Give, with proof, an example of a Banach space X where the map above $X \to X^{**}$ is not surjective.

Question 3.

a. State and prove the Banach-Steinhaus Theorem (you may assume any version of the Baire Category Theorem provided you state it clearly).

b. Let X, Y, Z be Banach spaces and suppose $B : X \times Y \to Z$ is a continuous bilinear form. Show there is some $M > 0$ such that $\|B(x,y)\| \leq M\|x\|\|y\|$ for all $x \in X, y \in Y$.