#1 $\left(\begin{array}{cccc|c} 1&3&1&2&1\\ 2&7&4&8&3\\ 1&4&3&6&1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1&3&1&2&1\\ 0&1&2&4&1\\ 0&1&2&4&0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1&0&-5&-10&-2\\ 0&1&2&4&1\\ 0&0&0&0&-1 \end{array}\right)$

This solution was revised.

No solution exists for this system

#2 $\left(\begin{array}{ccc|ccc} 2&1&1&1&0&0\\ 1&2&1&0&1&0\\ 1&1&2&0&0&1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1&1/2&1/2&1/2&0&0\\ 1&2&1&0&1&0\\ 1&1&2&0&0&1 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|ccc} 1&1/2&1/2&1/2&0&0\\ 0&3/2&1/2&-1/2&1&0\\ 0&1/2&3/2&-1/2&0&1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1&1/2&1/2&1/2&0&0\\ 0&1&1/3&-1/3&2/3&0\\ 0&1/2&3/2&-1/2&0&1 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|ccc} 1&0&1/3&2/3&-1/3&0\\ 0&1&1/3&-1/3&2/3&0\\ 0&0&4/3&-1/3&-1/3&1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1&0&1/3&2/3&-1/3&0\\ 0&1&1/3&-1/3&2/3&0\\ 0&0&1&-1/4&-1/4&3/4 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|ccc} 1&0&0&3/4&-1/4&-1/4\\ 0&1&0&-1/4&3/4&-1/4\\ 0&0&1&-1/4&-1/4&3/4 \end{array}\right) \quad \therefore A^{-1} = \frac{1}{4}\begin{pmatrix} 3&-1&-1\\ -1&3&-1\\ -1&-1&3 \end{pmatrix}$

#3 $\left(\begin{array}{cccc|c} 1&2&3&0&0\\ 0&1&1&0&0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1&0&1&0&0\\ 0&1&1&0&0 \end{array}\right)$.

$\therefore$ The solution space for the given system is

$\{(-x_3, -x_3, x_3, x_4) \in \mathbb{R}^4 \mid x_3, x_4 \in \mathbb{R}\}$

Hence its basis is $\{(-1,-1,1,0), (0,0,0,1)\}$.

#4 $\begin{pmatrix} 1&1&2&4\\ 1&2&2&5\\ 1&3&2&6 \end{pmatrix} \rightarrow \begin{pmatrix} 1&1&2&4\\ 0&1&0&1\\ 0&2&0&2 \end{pmatrix} \rightarrow \begin{pmatrix} 1&0&2&3\\ 0&1&0&1\\ 0&0&0&0 \end{pmatrix}$

Answer : $\begin{pmatrix} 1&0&2&3\\ 0&1&0&1\\ 0&0&0&0 \end{pmatrix}$

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#5 We directly do the QR decomposition. (Note that this is obtained by Gram-schmidtz orthogonalization process)

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{pmatrix}$$

$$A \cdot \begin{pmatrix} 1 & -2 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{pmatrix}$$

$$A \cdot \begin{pmatrix} 1 & -2 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 1 & 6 \end{pmatrix}$$

$$A \cdot \begin{pmatrix} 1 & -2 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A \cdot \begin{pmatrix} 1 & -2 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\therefore Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}$$

#6 Remark: In this problem, $b \in Ran(A)$, so the least square problem is actually same as directly solving $Ax = b$ (Instead of minimizing $\|Ax - b\|^2$) However, we will stick to the procedure that can be applied in more general setting of least square problem.

Method 1

Let $\vec{x} \in \mathbb{R}^3$ be a vector such that $\|A\vec{x} - \vec{b}\|^2$ is minimal. Then $\vec{b} - A\vec{x} \perp Ran A$. Hence $A^t(\vec{b} - A\vec{x}) = 0$. equivalently, we have to solve $(A^t A)\vec{x} = A^t \vec{b}$.

$$A^t A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 13 & 26 \\ 4 & 26 & 77 \end{pmatrix}$$

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Write $\vec{b} = (b_1, b_2, b_3)^t$

$$\Rightarrow A^t\vec{b} = \begin{pmatrix} b_1 \\ 2b_1+3b_3 \\ 4b_1+5b_2+6b_3 \end{pmatrix}$$

- Solving $(A^tA)\vec{x} = A^t\vec{b}$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 2 & 13 & 26 & 2b_1+3b_3 \\ 4 & 26 & 77 & 4b_1+5b_2+6b_3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 0 & 9 & 18 & 3b_3 \\ 0 & 18 & 61 & 5b_2+6b_3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 0 & 1 & 2 & b_3/3 \\ 0 & 18 & 61 & 5b_2+6b_3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1-2b_3/3 \\ 0 & 1 & 2 & b_3/3 \\ 0 & 0 & 25 & 5b_2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1-2b_3/3 \\ 0 & 1 & 2 & b_3/3 \\ 0 & 0 & 1 & b_2/5 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1-2b_3/3 \\ 0 & 1 & 0 & b_3/3-2b_2/5 \\ 0 & 0 & 1 & b_2/5 \end{array}\right)$$

$$\therefore \text{Solution} = \begin{pmatrix} b_1-2b_3/3 \\ b_3/3-2b_2/5 \\ b_2/5 \end{pmatrix}$$

Method 2 (Using QR decomposition that we found in $\boxed{\#5}$)

$$\begin{aligned} \|A\vec{x}-\vec{b}\| &= \|QR\vec{x}-\vec{b}\| \\ &= \|Q^tQR\vec{x}-Q^t\vec{b}\| \quad (\because Q : \text{orthogonal matrix}) \\ &= \|R\vec{x}-Q^t\vec{b}\| \end{aligned}$$

So its same as finding $\vec{x}$ that minimizes $\|Rx-Q^t\vec{b}\|$.

Since $Q^t\vec{b} \in \mathrm{Ran}(R)$, we will solve $R\vec{x} = Q^t\vec{b}$.

$$Q^t\vec{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \\ b_2 \end{pmatrix}$$

- Solving $R\vec{x} = Q^t\vec{b}$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 0 & 3 & 6 & b_3 \\ 0 & 0 & 5 & b_2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 0 & 1 & 2 & b_3/3 \\ 0 & 0 & 1 & b_2/5 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1-2b_3/3 \\ 0 & 1 & 2 & b_3/3 \\ 0 & 0 & 1 & b_2/5 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1-2b_3/3 \\ 0 & 1 & 0 & b_3/3-2b_2/5 \\ 0 & 0 & 1 & b_2/5 \end{array}\right)$$

$$\therefore \text{Solution} = \begin{pmatrix} b_1-2b_3/3 \\ b_3/3-2b_2/5 \\ b_2/5 \end{pmatrix}$$

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#7 Let $y = a_0 + a_1 x$ be the equation of line in $\mathbb{R}^2$ that we are looking for. Then (a_0, a_1) should be the solution that minimizes

$$\left\| \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} - \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} \right\|^2$$

This is the least square problem! As we have seen previously, we should solve $(A^t A)\vec{x} = A^t \vec{b}$ where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

$$A^t A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 8 & 26 \end{pmatrix}, \quad A^t \vec{b} = \begin{pmatrix} 36 \\ 112 \end{pmatrix}.$$

- Solving $(A^t A)\vec{x} = A^t \vec{b}$.

$$\left(\begin{array}{cc|c} 4 & 8 & 36 \\ 8 & 26 & 112 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 9 \\ 8 & 26 & 112 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 9 \\ 0 & 10 & 40 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 9 \\ 0 & 1 & 4 \end{array}\right)$$

$\therefore \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and the line of best fit is given by $y = 4x + 1$.

#8 Let $\vec{b} = (b_1, b_2, b_3)^t$. Let $A\vec{x}$ be projection of $\vec{b}$ onto $\mathrm{Ran}(A)$
$\Rightarrow (A^t A)\vec{x} = A^t \vec{b}$.

$$A^t A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad A^t \vec{b} = \begin{pmatrix} b_1 + b_3 \\ b_1 - b_2 \end{pmatrix}$$

- Solving $(A^t A)\vec{x} = A^t \vec{b}$.

$$\left(\begin{array}{cc|c} 2 & 1 & b_1 + b_3 \\ 1 & 2 & b_1 - b_2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1/2 & (b_1+b_3)/2 \\ 1 & 2 & b_1 - b_2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1/2 & (b_1+b_3)/2 \\ 0 & 3/2 & b_1/2 - b_2 - b_3/2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 0 & b_1/3 + b_2/3 + 2b_3/3 \\ 0 & 1 & b_1/3 - 2b_2/3 - b_3/3 \end{array}\right)$$

Write $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1/3 + b_2/3 + 2b_3/3 \\ b_1/3 - 2b_2/3 - b_3/3 \end{pmatrix}$. Then $A\vec{x} = \begin{pmatrix} x_1 + x_2 \\ -x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 2b_1/3 - b_2/3 + b_3/3 \\ -b_1/3 + 2b_2/3 + b_3/3 \\ b_1/3 + b_2/3 + 2b_3/3 \end{pmatrix}$

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Hence the projection matrix is $P = \begin{pmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{pmatrix}$

#9 Applying elementary row operation on A yields.

$$A = \begin{pmatrix} 0 & 0 & 1 & -2 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 0 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 2 & 2 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In the last form, there are three pivot variable. So $rk(A)=3$

#10

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\therefore P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}.$$

$$U = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

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a. False.

Let $A \in Mat_{4,9}(\mathbb{R})$ and Suppose $\dim(\ker A) = \dim(\operatorname{Ran} A) = r$.

Then by dimension theorem $9 = 2r$. $(\rightarrow\leftarrow)$

b. False.

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ has independent rows, but $N(A) = \langle (0,0,1) \rangle \neq \{0\}$

c. False.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \in Mat_{3,2}(\mathbb{R}) \Rightarrow C(A)$ is 2-dimensional subspace of $\mathbb{R}^3$.

d. False.

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow C(A) = N(A) = \langle (1,0) \rangle$, not orthogonal.

e. True

$A \in Mat_{m,n}(\mathbb{R}) \Rightarrow A^t \in Mat_{n,m}(\mathbb{R}) \Rightarrow C(A^t) \subseteq \mathbb{R}^n$

f. False

$\mathbb{R}^{100}$ is 100-dimensional. Any set consisting of r vectors with $r > 100$ is linearly dependent.

g. True

We can complete any such vectors to be basis. But $\dim \mathbb{R}^{100} = 100 \Rightarrow$ [This set is already a basis]

h. True.

Let S' be collection of 100 vectors. Then we can find $S \subseteq S'$.

such that $\begin{cases} S \text{ is linearly independent} \\ S \text{ spans } \mathbb{R}^{100} \end{cases}$

In other words, S is basis. Since $\dim(\mathbb{R}^{100}) = 100$, $S = S'$ and hence S' is already independent.

i. False.

Let $v_i = (0,0,i)$ for $i = 1,2,3$. Then $\langle v_1, v_2, v_3 \rangle = \langle (0,0,1) \rangle \neq \mathbb{R}^3$

j. False

$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad A' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

here A' is a row reduced echelon form of A.

But $C(A) = \langle (1,1)^t \rangle$, $C(A') = \langle (1,0)^t \rangle \Rightarrow C(A) \neq C(A')$

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k False

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

A does not admit LU decomposition. By contradiction, suppose

$$A = \begin{pmatrix} l_{11} & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \begin{pmatrix} u_{11} & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} = LU.$$

with some lower triangular L and upper triangular U.
write (i,j)-th entry of L to be l_{ij} and similarly u_{ij} for U.
Note that $l_{11}u_{11} = (1,1)$-th entry of $A = 0 \Rightarrow l_{11} = 0$ or $u_{11} = 0$.

i) $l_{11} = 0 \Rightarrow 1^{st}$ row of A should be 0 vector. $(\rightarrow\leftarrow)$

ii) $u_{11} = 0 \Rightarrow 1^{st}$ column of A should be 0 vector $(\rightarrow\leftarrow)$

l True

Using special solution, each free column is a linear combination of former pivot column. $\Rightarrow$ Pivot columns span column space. To see that these are independent, note that dimension of column space is equal to rank of matrix, which is again equal to number of pivot column.

m False

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \in Mat_{3,2}(\mathbb{R}) \qquad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow b \notin Ran(A)$$

So $Ax = b$ has no solution. but $rk(A) = 2$.

n True

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a row of A, and it is in $\ker A$.

o. False.

Let $V = \{(x,0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ and P be a projection of $\mathbb{R}^2$ onto V. Let $v = (1,1)$ then $Pv = (1,0)$ but $\langle Pv, v\rangle = 1 \neq 0$.

p True.

$V_1 = \mathbb{R}^4 \times \{0\} \subseteq \mathbb{R}^8$, $V_2 := \{0\} \times \mathbb{R}^4 \subseteq \mathbb{R}^8 \Rightarrow V_1 \perp V_2$

q True.

Let $A \in Mat_{m,n}(\mathbb{R})$ and consider the least square problem minimizing $\|A\vec{x} - \vec{b}\|^2$. We can write $\mathbb{R}^m = Im(A) \oplus Im A^\perp$. and hence $\vec{b} = A\vec{x}_0 + \vec{c}$ for some $\vec{x}_0 \in \mathbb{R}^n$. $A\vec{x}_0$ is the point minimizing $\|\vec{b} - A\vec{x}\|^2$. (Although $\vec{x}_0$ is not necessarily unique)

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r. True.

$\vec{b} \in C(A) \Rightarrow$ by definition, $\vec{b} = \sum_{i=1}^{n} b_i(A\vec{e_i})$ for some $b_i \in \mathbb{R}$.

$\Rightarrow \vec{x} = \sum b_i \vec{e_i}$ is a solution to $A\vec{x} = \vec{b}$.

s. True

$V^\perp = \{w \in \mathbb{R}^n \mid \langle v, w \rangle = 0 \text{ for } \forall v \in V\}$ is uniquely determined.

t. True.

Let $P \in Mat_{n,n}(\mathbb{R})$ be a permutation matrix. Then there exists a bijection $\sigma : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ such that $Pe_i = e_{\sigma(i)}$

Now $\langle Pe_i, Pe_j \rangle = \langle e_{\sigma(i)}, e_{\sigma(j)} \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$

u. False

$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is permutation matrix which is not symmetric.

v. True

$(A^t A)^t = A^t (A^t)^t = A^t A.$

w. True.

$BA = I \Rightarrow Ax = 0$ has only trivial solution ($\because$ multiply by B on the left)

$\Rightarrow$ Reduced row echelon form of A is $Id_{n,n}$

$\Rightarrow A$ is product of elementary matrices.

$\Rightarrow A$ is invertible (i.e. $\exists A^{-1} \in Mat_{n,n}(\mathbb{R})$ such that $AA^{-1} = A^{-1}A = I$)

$\Rightarrow B = A^{-1}$ and so $AB = I$.

x. True.

Let $v_1, \cdots, v_n$ be orthonormal vectors. Suppose $\sum_{i=1}^{r} a_i v_i = 0$ Then

$\langle \sum_{i=1}^{r} a_i v_i, v_j \rangle = a_j \langle v_j, v_j \rangle = a_j = 0$ for $\forall j \in \{1, \cdots, r\}$

y. True

$A \in Mat_{n,n}(\mathbb{R})$ is invertible

$\Rightarrow Ax = 0$ has only trivial solution

$\Rightarrow Ae_1, \cdots, Ae_n$ are linearly independent.

z. False.

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$

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$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Id_{2,2}, \quad BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq Id_{3,3}$