Functional Analysis (325b/525b) Problem Set 1

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Question 1.

Let

$$A_n = \left\{ x = (x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i = n,\ x_i \geq 0\ \forall i \right\}.$$

1. Show that $g(x) = \prod_{i=1}^{n} x_i$ is bounded on A_n and attains its supremum at some $z = (z_1, \ldots, z_n) \in A_n$.
2. Suppose $x \in A$ and $x_1 = \min x_i < x_2 = \max x_i$. Set $y_1 = y_2 = \frac{1}{2}(x_1 + x_2)$ and $y_i = x_i$ for $3 \leq i \leq n$. Show that $y = (y_1, \ldots, y_n) \in A_n$ and $g(y) > g(x)$. Deduce $z_i = 1$ for all i.
3. Deduce the AM-GM inequality.

Question 2.

Prove the rearrangement inequality: if

$$x_1 \leq x_2 \leq \ldots \leq x_n, \quad y_1 \leq y_2 \leq \ldots \leq y_n$$

are real numbers, then for any permutation σ of the set $\{1, \ldots, n\}$,

$$x_1 y_n + x_2 y_{n-1} + \ldots + x_n y_1 \leq x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \ldots + x_n y_{\sigma(n)} \leq x_1 y_1 + \ldots + x_n y_n.$$

Also show that if the x_i are distinct and the y_i are distinct, then the only permutations for which either the lower bound or upper bound becomes an equality are the obvious ones (the order reversing permutation and the identity, respectively).

Question 3.

Show that if $a_1, a_2, \ldots, a_n$ are positive and $b_i = a_{\sigma(i)}$ for some permutation σ of the set $\{1, \ldots, n\}$ then

$$\sum_{i=1}^{n} \frac{a_i}{b_i} \geq n.$$

Question 4.

Let $f : \mathbf{R} \to \mathbf{R}$ be a convex function. Show that

$$F(x_1, x_2) \equiv \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

is monotonically non-decreasing in x_1 for each fixed x_2. Deduce that for any closed bounded interval $I = [a, b]$ there is a constant K such that

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$$

for all $x_1, x_2 \in I$. This is called *Lipschitz continuity.* Deduce that f is continuous.

Question 5.

Show that in a normed space, the closure of the open unit ball (about the origin) is the closed unit ball about the origin. Is this statement true for every metric space?

Question 6.

Show that ℓ_p is a Banach space for each $1 \leq p \leq \infty$. Also show that c_0, the space of sequences tending to zero, is a closed subspace of ℓ_∞. Show $\ell_p (1 \leq p < \infty)$ are separable, while ℓ_∞ is not.

Question 7.

For $x \in \ell_1$ set

$$\|x\|' = 2\left|\sum_{n=1}^{\infty} x_n\right| + \sum_{n=2}^{\infty}\left(1 + \frac{1}{n}\right)|x_n|.$$

Show that $\| \bullet \|'$ is a norm on ℓ_1 and that ℓ_1 is complete with respect to this norm. Is this norm equivalent to the standard ℓ_1 norm?

Question 8.

Show that on every infinite dimensional normed space V there exists a discontinuous linear functional: i.e. a linear map $\phi : V \to \mathbf{R}$ that is not continuous.