Functional Analysis (325b/525b) Problem Set 2

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Question 1.

Let p and q be conjugate indices, with $1 \leq p < \infty$. Prove that $\ell_p^* = \ell_q$. Show also that $c_0^* = \ell_1$.

Question 2.

Let c be the subspace of ℓ^∞ consisting of convergent sequences. What is the general form of a bounded linear functional on c?

Question 3.

Check that for $1 < p < \infty$, the space ℓ_p is reflexive. Check also that ℓ_1, ℓ_∞ and c_0 are not reflexive.

Question 4.

Let X be a Banach space. Show that if $X^{***} = X^*$ then X is reflexive, i.e. $X^{**} = X$.

Question 5.

Let K be a closed convex set in a real normed space X. Show that every boundary point of K has a support functional: for every $x_0 \in \delta K$ there is an $f \in X^*$ such that $f \neq 0$ and $\sup_{x \in K} f(x) = f(x_0)$.

Question 6.

Let $x_1, \ldots, x_n$ be non zero elements of a normed space X. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{R}^n$ set

$$\|\lambda_0\| = \max\left\{ \|\sum_{i=1}^{n} \epsilon_i \lambda_i x_i\| \;:\; \epsilon_i \in \{\pm 1\} \text{ for } i = 1, \ldots, n \right\}.$$

Show that $\| \bullet \|_0$ is a norm on $\mathbf{R}^n$. Show also that for $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $j = (1, 1, \ldots, 1)$ that

$$\|\lambda\|_0 = \|j_0\| \max_i |\lambda_i|.$$

Question 7.

Let Y be a finite dimensional subspace of an infinite dimensional normed space X. Show that for every $\epsilon > 0$ there is an x in X with $\|x\| = 1$ such that $\|y\| \leq (1 + \epsilon)\|x + y\|$ for every $y \in Y$.

Question 8.

Show that a Banach space cannot have a countably infinite algebraic (Hamel) basis. Hint: use Corollary 9 from the textbook.