Functional Analysis (325b/525b) Problem Set 3

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A (+) denotes an extra hard question.

Question 1.

Let $f : (0, \infty) \to \mathbf{R}$ be a continuous function such that $\lim_{n\to\infty} f(nx) = 0$ for all $x > 0$. Show that $\lim_{x\to\infty} f(x) = 0$.

Question 2+.

Let $f : \mathbf{R} \to \mathbf{R}$ be smooth (infinitely differentiable). Suppose that for all x there is a k_x such that $f^{(k)}(x) = 0$ for all $k \geq k_x$. Prove that f is a polynomial.

Question 3.

Let $\phi_n : [0, 1] \to (0, \infty)$ be uniformly bounded continuous functions such that

$$\int_0^1 \phi_n(x)dx \geq c$$

for some $c \geq 0$ and all n. Suppose $c_n \geq 0$ for all $n = 1, 2, 3, \ldots$ and

$$\sum_{n=1}^{\infty} c_n \phi_n(x) < \infty$$

for every $x \in [0, 1]$. Prove that

$$\sum_{n=1}^{\infty} c_n < \infty.$$

Question 4.

Let X be a normed vector space and $S \subset X$. Show that if

$$\{ f(x) : x \in S \}$$

is bounded for every $f \in X^*$ then S is bounded, that is, there is $K > 0$ such that $\|x\| \leq K$ for all $x \in S$.

Question 5.

Show that a metric space is compact if and only if it is totally bounded and complete.

Question 6.

Show that a subset of a complete metric space is totally bounded if and only if its closure is compact.

Question 7.

Let U be an open subset of $\mathbf{C}$ and let $f_1, f_2, \ldots : U \to \mathbf{C}$ be uniformly bounded analytic functions. Prove that there is a subsequence f_{n_k} that is uniformly convergent on every compact subset of U.

Question 8.

Let G be an open subset of $\mathbf{R}^2$ and let $f : G \to \mathbf{R}$ be continuous. Use the Arzelà-Ascoli Theorem to prove Peano's theorem that for each point $(x_0, y_0) \in G$, at least one solution of

$$y'(x) = f(x, y)$$

passes through (x_0, y_0).

Hint: Let V be a closed neighborhood of (x_0, y_0) and set $K = \sup\{|f(x, y)| \ : \ (x, y) \in V\}$. Choose $a < x_0 < b$ such that

$$\left\{(x, y) : x \in [a, b], x \neq x_0, \ \frac{|y - y_0|}{|x - x_0|} < K\right\} \subset U.$$

The aim is to show that our differential equation has a solution through (x_0, y_0) defined on $[a, b]$. Find such a solution by considering piecewise linear approximations, say with division points $x_0 \pm k/n, \ k = 1, 2, \ldots$.