Functional Analysis (325b/525b) Problem Set 4

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Question 1.

Show that a contraction of an incomplete metric space into itself need not have a fixed point.

Question 2.

Show that if f is a map of a complete metric space X into itself such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ $(x \neq y)$ then f need not have a fixed point. Show that if X is compact then f does have a unique fixed point.

Question 3.

Show that

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|$$

defines a metric on ℓ_∞ and the restriction of this metric to the closed unit ball $\overline{B}_{\ell_\infty}$ induces the weak star topology on $\overline{B}_{\ell_\infty}$.

Question 4.

Let X be an infinite dimensional normed vector space. Show that the weak closure of

$$S(X) = \{x \in X \;:\; \|x\| = 1\}$$

is

$$\overline{B_X} = \{x \in X \;:\; \|x\| \leq 1\}.$$

Question 5.

Let X be a normed vector space. Show that the sets

$$U(\phi_1, \ldots, \phi_N; \epsilon_1, \ldots, \epsilon_N; x) = \{y \in X \;:\; |\phi_j(y) - \phi_j(x)| < \epsilon_j \;,\; j = 1, \ldots, n\}$$

for $\phi_j \in X^*, \epsilon_j > 0$ are a local basis at x for the weak topology on X.

Question 6.

Let X be an infinite-dimensional Banach space whose unit ball has only finitely many extreme points. Show there can not be any normed vector space Y so that $X = Y^*$.