Research Statement

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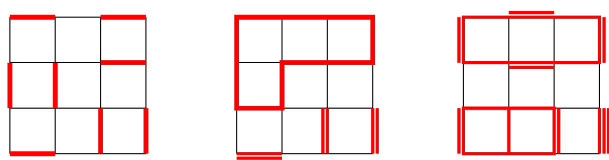
My research is mainly in the area of combinatorics, and particularly what are called *dimer models*. This is the study of enumeration of perfect matchings of a graph, and associated probability measures on the set of perfect matchings. I study various ways in which dimer models relate to other topics, such as combinatorics, algebra, representation theory, symplectic and Poisson geometry, integrable systems, mathematical physics, and statistical mechanics. My work is closely related to *cluster algebras*, and various algebraic and geometric objects which have a cluster algebra structure, such as Grassmannians, Teichmüller spaces, rings of invariants of algebraic groups, and various integrable dynamical systems.

Below, I will first briefly describe the setup of the dimer model, and then mention some of my published results involving dimer models, their generalizations, and their appearance in different areas (particularly those related to cluster algebras). Throughout, I will mention some current ongoing work, and possible future plans.

Dimer Models

Given a bipartite planar graph G, the *dimer model* on G is the study of random perfect matchings of G. A *perfect matching* (also called a *dimer cover*) is a subset of edges of the graph such that every vertex is incident to exactly one edge. An example of a perfect matching is pictured below, on the left (the edges of the matching are in red):



dimer cover 2-dimer cover 3-dimer cover

Suppose the edges of G are weighted by positive real numbers. The primary enumerative quantity in the dimer model is the *partition function*

$$Z := \sum_{\text{dimer cover } M} \mathrm{wt}(M)$$

where $\mathrm{wt}(M)$ is the product of the edge weights occuring in M. It was shown by Kasteleyn in the '60s [Kas63] that the partition function is the determinant of a modified version of the adjacency matrix of the graph (which we now call the *Kasteleyn matrix*, K). This matrix also allows to compute certain probabilistic quantities, such as the probability of a certain edge appearing in a random dimer cover.

Higher Dimer Models

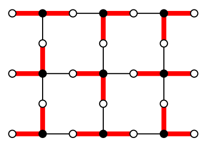
Informally, an n-*dimer cover* is the result of superimposing n perfect matchings. More technically, it is a multiset of edges such that every vertex is incident to n edges. Examples for $n = 2$ and $n = 3$ are pictued in the middle and right of the above figure. Some of my recent work has focused on different types of n-dimer models, and their relations to various topics such as representation theory, lattice models in statistical mechanics, cluster superalgebras, and higher-dimensional continued fractions. I will present a few of these results below.

In recent work with Richard Kenyon, we considered *mixed dimer models*, which are even more general than n-dimer models. In this model, every vertex v of a graph is assigned an integer n_v, and we consider multisets of edges such that each v is incident to n_v edges. The n-dimer model is the special case where all vertices are assigned the same value n. Also, as part of the data, each edge with endpoints b and w is labelled by an $n_w \times n_b$ matrix and these matrices are used to define the weights/probabilities in the model. When all vertex multiplicities are the same value n, this data is equivalent to a GL_n*-local system* (which is a GL_n-representation of the fundamental group of the graph). We call these probability measures on n-dimer covers *higher rank dimer models* or the GL_n*-dimer model*. We showed that a version of Kasteleyn's theorem is true in this setting:

Theorem 1 (Kenyon, Ovenhouse [KO23]). *There is a matrix K (which is a block matrix analog of the usual Kasteleyn matrix) such that the partition function of the mixed dimer model is given by $Z = |\det(K)|$.*

There is a theorem of Kenyon which says that in the single dimer model, the probability that a given edge appears in a random dimer cover is given by an entry of the inverse Kasteleyn matrix. It is also possible to express the covariance between different edge probabilities directly in terms of the inverse Kasteleyn matrix. In current ongoing work with a group of graduate students, which began at a research workshop at the University of Minnesota, we are also able to give analogous formulas in the higher rank dimer models in terms of the block Kasteleyn matrix mentioned in Theorem 1.

It is a simple observation that many lattice models in statistical mechanics (such as the "*free fermionic*" 6-vertex and 20-vertex models) are equivalent to certain mixed dimer models. For instance, the 6-vertex model on a 3×3 grid with *domain wall boundary conditions* is equivalent to a mixed dimer model on the graph pictured below, where black vertices have degree 2 and white vertices have degree 1. In the picture, an example configuration is given by the red edges.



As such, our result gives a new way of seeing that their partition functions are determinantal for certain choices of weights. The second main result in the aforementioned work is that to each n-dimer model or mixed dimer model on a graph G, there exists another edge-weighted planar graph $\widetilde{G}$ whose single-dimer model is equivalent to the n-dimer or mixed dimer model on the original graph. The graph $\widetilde{G}$ is obtained from G by replacing each vertex with some graph, and replacing each edge of G by several parallel edges.

Theorem 2 (Kenyon, Ovenhouse [KO23]). *There is a many-to-one weight preserving map from single dimer covers of $\widetilde{G}$ to n-dimer covers (or mixed dimer covers) of G with a matrix connection. This means the weight of a mixed dimer cover of G is equal to the sum of the weights of all single dimer covers of $\widetilde{G}$ which are its pre-images.*

This result allows questions about mixed dimer models to be translated to questions about single dimer models, which are more well-understood. It also relates the GL_n-dimer models to certain cluster algebra structures, as the details of passing from G to $\widetilde{G}$ involve the cluster parameterization of the totally positive Grassmannian.

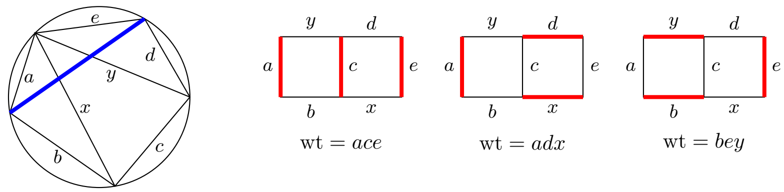
Consider the following *Deligne-Simpson problem*: find matrices A and B, such that A, B, and AB have certain precscribed eigenvalues. In [KO24], Rick Kenyon and I used these ideas (coming from Thorem 2) to give a cluster-like parameterization of the set of solutions to this problem. We found a certain honeycomb graph drawn on the torus such that the solution space of the Deligne-Simpson problem can be identified with the space of edge weights of this graph.

I am also working on a project with Rick Kenyon, Dan Douglas, Sam Panitch, and Sri Tata involving a quantum version of the GL_n-dimer model. It is a generalization of the quantum SL_n-invariants coming from webs. Webs are planar graphs which encode fundamenal SL_n-invariant functions on certain representations. We are able to write the quantum partition function of this model as an expression which is similar to the (quantum) determinant of the associated Kasteleyn matrix. In the case $n = 2$, we are also able to extract certain combinatorial statistics in the double-dimer model using the quantum version.

Cluster Superalgebras and Teichmüller Theory

The decorated Teichmüller space of a bordered, punctured surface is the parameter space of hyperbolic structures on that surface. It has coordinate charts corresponding to ideal triangulations, with coordinates given by "λ*-lengths*", which are the lengths of truncated geodesics. Choosing a different triangulation gives a change-of-coordinate map, which is a *mutation* in a suitably-defined *cluster algebra*. The expressions for the new coordinates, in terms of the old, are given by partition functions of a dimer model on a certain planar graph.

For a simpler Euclidean (as opposed to hyperbolic) example, consider a pentagon inscribed in a circle, with a chosen triangulation, as in the figure below. The edge labels are the lengths.



By using Ptolemy's theorem, the length of the blue diagonal can be expressed as $\frac{ace+adx+bey}{xy}$, the numerator of which is the weighted sum of all three dimer covers of the graph pictured in the right of the figure.

For a surface S and a Lie group G, the *higher Teichmüller space* is the space of G-representations of the fundamental group of S (up to equivalence). With co-authors Gregg Musiker and Sylvester Zhang, we studied the higher Teichmüller space for the ortho-symplectic Lie supergroup $\mathrm{Osp}(1|2)$, and we proved an extension of the fact mentioned above, which establishes a version of the celebrated *Laurent phenomenon* for these cluster superalgebras.

Theorem 3 (Musiker, Ovenhouse, Zhang [MOZ22])**.** *Given two coordinate charts on the decorated super Teichmüller space of a marked disk, the even coordinates in the new chart, when expressed in terms of the old coordinates, are given by partition functions for a double dimer model on a certain planar graph. In particular, the super cluster mutations obey a Laurent phenomenon analogous to ordinary cluster algebras.*

Integrable Systems from Dimer Models

A dynamical system is *integrable* if it preserves some symplectic/Poisson structure, and admits sufficiently many conserved quantities. Goncharov and Kenyon introduced a class of integrable systems coming from the dimer model of graphs drawn on a torus. In these models, the conserved quantities (called the *Hamiltonians*) are the weighted sums of dimer covers corresponding to a fixed homology class on the torus.

With Semeon Arthamonov and Michael Shapiro, we studied a noncommutative version of these systems, and demonstrated their integrability by proving a noncommutative analog of the R-matrix formula for the Poisson brackets (R-matrices are a ubiquitous tool in the theory of integrable systems).

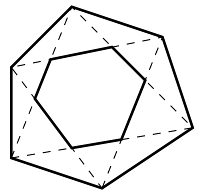
Theorem 4 (Arthamonov, Ovenhouse, Shapiro [AOS24])**.** *For a graph drawn on a torus, the noncommutative Poisson structure is given by a suitable generalization of the classical R-matrix formula:*

$$\{M(\lambda), M(\mu)\} = [M(\lambda) \otimes M(\mu),\ R(\lambda, \mu)]$$

From this, it can be shown that the (noncommutative analogs of the) dimer partition functions Poisson-commute, forming an integrable system.

The main utility of the noncommutative Poisson structures is that they induce ordinary Poisson structures on character varieties (i.e. higher Teichmüller spaces). As a corollary of our result, we thus conclude that the dimer integrable systems lift to integrable systems on the higher rank GL_n-dimer models.

A fun and interesting application of these results is given by a generalization of the *pentagram map*. The pentagram map is a dynamical system on the space of polygons drawn in the plane, given by a very elementary construction. One draws all "short diagonals" (connecting vertices with distance 2), and the image of the map is the smaller polygon formed by the intersections of these diagonals. An illustration is given below:



The pentagram map is known to be integrable. Since its introduction, many generalizations have been studied, some integrable and some not. One particular generalization (introduced by Felipe and Marí-Beffa) has the polygons living in a Grassmannian manifold. In my PhD thesis [Ove20], I related this Grassmannian pentagram map to the noncommutative dimer integrable systems mentioned above, and showed its integrability via early verions of the results from [AOS24].

Higher Continued Fractions and Dimer Models

The planar graphs mentioned in Theorem 3 are what are called "*snake graphs*", which are equivalently certain kinds of skew Young diagrams called "*border strips*" or "*ribbons*". They consist of a sequence of square faces, each attached to the previous on either the right or top edge.

Inspired by Theorem 3, we considered the purely combinatorial problem of counting n-dimer covers on snake graphs. From the $n = 1$ case, it was known that there is an association between snake graphs and continued fractions. A rational number $x = [a_1, a_2, \dots, a_n]$ determines a snake graph G_x (whose shape is related to the integers $a_1, \dots, a_n$). A theorem of Shiffler and Canakci says that if $x' = [a_2, a_3, \dots, a_n]$, then

$$x = \frac{\text{\# of perfect matchings of } G_x}{\text{\# of perfect matchings of } G_{x'}}$$

For a given n, let A be the $(n+1) \times (n+1)$ upper-triangular matrix where all entries on or above the diagonal are equal to 1, and let $B := A^\top$. Then we have the following.

Theorem 5 (Musiker, Ovenhouse, Schiffler, Zhang [MOSZ23]). *Let G_x be a snake graph with associated continued fraction $x = [a_1, a_2, \dots, a_{2k}]$, and let M be the matrix*

$$M := A^{a_1} B^{a_2} A^{a_3} B^{a_4} \cdots A^{a_{2k-1}} B^{a_{2k}}$$

Then the upper-left entry M_{11} is the number of n-dimer covers of G_x.

We also gave combinatorial interpretations for *all* entries of the matrix M, and we gave bijections between n-dimer covers and other classical combinatorial objects such as north-east lattice paths, order ideals of posets, and plane partitions or P-partitions.

This led us to define certain types of *higher continued fractions* which are related to n-dimer models in an analogous way. In particular, for each positive integer m, and each $i \leq m$, we defined a higher continued fraction map $r_{i,m} \colon \mathbb{Q} \to \mathbb{Q}$ such that

$$r_{m,m}(x) = \frac{\text{\# of } m\text{-dimer covers of } G_x}{\text{\# of } m\text{-dimer covers of } G_{x'}}$$

with similar interpretations for the other $r_{i,m}$ with $i < m$. We showed that these higher continued fraction values have a recursive formula generalizing the usual reccurence for ordinary continued fractions.

A pleasant property of these higher continued fractions is that the definition extends to irrational values as well.

Theorem 6 (Musiker, Ovenhouse, Schiffler, Zhang [MOSZ23]). *If x is an irrational number with infinite continued fraction $x = [a_1, a_2, a_3, \dots]$, and $x_n = [a_1, \dots, a_n]$ are its rational approximations, then the limits $\lim\limits_{n \to \infty} r_{i,m}(x_n)$ exist for all i, m. Furthermore, if x is a quadratic irrational (i.e. its continued fraction is eventually periodic), then $r_{i,m}(x)$ is an algebraic number of degree $m + 1$.*

The continued fractions of quadratic irrationals are eventually periodic integer sequences. *Hermite's problem* asks whether it is possible to encode real numbers by integer sequences (similar to the continued fraction expansion) such that a number is a cubic irrational precisely when its corresponding sequence is eventually periodic. Our theorem above says that the m-continued fraction encoding maps eventually perioidic sequences to degree $m+1$ algebraic numbers. It is an interesting open problem to explore whether the converse is true: Does the m-continued fraction of every algebraic number give an eventually periodic sequence?

I will also mention that these higher continued fractions have led to some successful research projects with undergraduates. In particular, two undergraduates at Yale helped me investigate some unanswered questions from our original paper:

Theorem 7 (Basser, Ovenhouse, Sakarda [BOS24])**.** *The higher continued fraction maps $r_{i,m} : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and monotone increasing.*

I will conclude this section by mentioning some interesting work-in-progress on this topic. Together with Andrew Claussen, we are investigating the enumeration of mixed dimer covers on snake graphs, where different vertices have different degrees. In some natural simple cases, we have discovered some remarkable appearances of famous combinatorial sequences, including the Euler and Catalan numbers. In particular, we are considering the straight and zigzag snake graphs (with shapes $RRR\cdots$ and $URUR\cdots$), with vertices of multiplicities $1, 2, 3, 4, \dots$, from left-to-right. We then have the following.

Theorem 8 (Claussen, Ovenhouse)**.**

(a) The number of mixed dimer covers on the straight snake graph with $n-2$ squares is the euler number E_n (the number of alternating permutations in S_n).

(b) The number of mixed dimer covers on the zigzag snake graph with $n-1$ squares is the Catalan number C_n.

Moreover, we have explicit bijections between mixed dimer covers and the corresponding sets of permutations, such that the partial order on mixed dimer covers maps to a partial order on permutations which is a coarsening of the Bruhat order on the symmetric group.

References

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