Research Statement
Nicholas Ovenhouse

My research is mainly in the area of algebraic combinatorics, with connections to Poisson geometry and dynamical systems. Specifically, I am interested in problems related to, and inspired by, the theory of cluster algebras. Below are summaries of some of my published results, along with some background. Possible directions of future research are also discussed.

Cluster Algebras

Cluster algebras are special kinds of commutative algebras. They are subrings of a fraction field $\mathbb{Q}(x_1, \ldots, x_n)$ whose generators are defined by a recursive combinatorial procedure called mutation [FZ02]. One of the earliest and most basic results in the theory is what is known as the Laurent Phenomenon. It says that each of the generators (called cluster variables), which are, by definition, rational functions, are in fact always Laurent polynomials. It is often an interesting problem to explicitly describe, combinatorially, the terms that occur in these Laurent polynomial expressions.

One interesting class of cluster algebras come from two-dimensional triangulated surfaces [FST08] [FG06]. The cluster variables represent coordinate functions on the decorated Teichmüller space of the surface [Pen12] [FT18], and mutations are certain transition function between coordinate charts. For this cluster structure, several combinatorial interpretations of the Laurent polynomial expansions are known [Sch08] [Yur19] [MSW11] [MW13].

My Contributions

There has been a trend in recent years to try to find a suitable definition of cluster super-algebra (i.e. a super-commutative algebra generalizing the notion of cluster algebra). A super-commutative algebra is an algebra $A = A_0 \oplus A_1$ with a $\mathbb{Z}_2$-grading, such that elements of $A_0$ are central, and elements of $A_1$ anti-commute with each other (i.e. $xy = -yx$ for $x, y \in A_1$).

Several different approaches have been suggested (e.g. [MGOT15], [Ovs15], [OS18] [LMRS21], [SV22], [She22]), but all of them are slightly different from one another, and each seems to be specially tailored to a specific example and application. So far there is no general definition that encompasses them all.

Recently, a super-manifold version of the decorated Teichmüller space was introduced by Penner and Zeitlin [PZ19], where the authors gave a super-version of the Ptolemy relation for hyperbolic geodesic lengths. In this setup, there is a super-algebra with even generators corresponding to the diagonals of a fixed triangulation, and odd generators corresponding to the triangles.

In joint work with Gregg Musiker and Sylvester Zhang [MOZ21] [MOZ22a] [MOZ22b], we have proved several results indicating that this decorated super Teichmüller space enjoys many of the cluster-algebraic properties of the ordinary (non-super) decorated Teichmüller space. As mentioned above, one of the most striking and powerful properties of cluster algebras is the Laurent phenomenon. One of our main results is that this is still the case, in some sense, for the super case:

**Theorem 1.** [MOZ21, MOZ22a] (Laurent Phenomenon for Decorated Super Teichmüller Space)

Let $P$ be a polygon with vertices 1, 2, ..., $n$, and a chosen triangulation. Let $\lambda_{ij}$ and $\mu_{ijk}$ be the coordinates of the decorated super Teichmüller space. For any diagonal $(a, b)$ which is not in the triangulation, the variable $\lambda_{ab}$ is a polynomial in the $\mu_{ijk}$'s, and a Laurent polynomial in the square roots of the $\lambda_{ij}$'s.
We actually proved much stronger versions of Theorem 1. We gave several specific descriptions of these Laurent polynomials as combinatorial generating functions, generalizing known formulas in the ordinary (non-super) case.

In [Sch08], it was shown that $\lambda_{ij} = \sum_{p \in T_{ij}} \text{wt}(p)$ where the sum is over the set of “$T$-paths”. In [MOZ21], we define the notion of “super $T$-paths”, and prove that the super $\lambda$-lengths have a similar interpretation as a weight generating function.

In [MSW11], it was shown that for each $(i, j)$, there is some planar graph $G$ so that $\lambda_{ij} = \frac{1}{\text{cross}(i, j)} \sum_{M \in D(G)} \text{wt}(M)$, where $\text{cross}(i, j)$ is the product of $\lambda$-lengths of the diagonals crossed by $(i, j)$, and $D(G)$ is the set of “perfect matchings” (or “dimer covers”) of $G$. In [MOZ22a], we showed that the super $\lambda$-lengths are given by the same formula, but where the sum is over double dimer covers of the same graph $G$, for an appropriate notion of the “weight” of a double dimer cover.

Poisson Geometry and Integrable Systems

An associative, commutative algebra $A$ is called a Poisson algebra if it is also a Lie algebra (with Lie bracket denoted $\{-,-\}$) subject to the “Leibniz identity”, which relates the Lie and associative products:

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}$$

Poisson algebras often occur as algebras of functions on some manifold or algebraic variety, in which case the space is called a Poisson manifold or Poisson variety.

Let $T: M \to M$ be a Poisson mapping on a Poisson manifold $M$. Repeated application of $T$ may be considered as a dynamical system with discrete time. The map $T$ is said to be “completely integrable” if there exist functions $f_1, \ldots, f_k \in C^\infty(M)$ ($k$ is half of the “rank” of the Poisson bracket) which are invariant under $T$, and such that $\{f_i, f_j\} = 0$ for all $1 \leq i, j \leq k$.

I usually consider Poisson structures and integrable systems which are related to cluster algebras. In particular, there are certain Poisson structures which occur frequently and naturally alongside cluster structures. Also, there are many examples of discrete dynamical systems which can be realized as a sequence of cluster mutations, and the cluster structure usually helps to see that such systems are integrable.

My Contributions

Consider a network (weighted directed graph) embedded on a cylinder, with some source vertices on one boundary circle, and some sink vertices on the other. An example is pictured below:

\[
B(\lambda) = \begin{pmatrix}
dc & d\lambda \\
bd + adc & ad\lambda
\end{pmatrix}
\]

One may then consider the space of all choices of edge weights, which is just $\mathbb{R}^N$ (where $N$ is the number of edges). There is a relatively simple Poisson structure one can define on this space [GSV12], which is log-canonical in the sense described earlier.

To such networks one can associate a path-weight matrix $B(\lambda)$, whose entry $b_{ij}(\lambda)$ is a Laurent polynomial which is the sum of all path weights from source $i$ to sink $j$, and powers of $\lambda$ record the winding number around
the cylinder. It was proven in [GSV12] that the Poisson brackets between entries of this matrix satisfy the Sklyanin $R$-matrix bracket (for the trigonometric $R$-matrix associated to $\text{SL}_n$). This is expressed in the formula:

$$\{ B(\lambda), B(\mu) \} = [R(\lambda, \mu), B(\lambda) \otimes B(\mu)]$$

From this expression, one can show that the collection of spectral invariants (i.e. the coefficients of the polynomials $\text{tr}(B(\lambda)^k)$) Poisson-commute, giving an integrable system.

In recent work with Semeon Arthamonov and Michael Shapiro [AOS20], we generalized these results to the case where the edge weights on the graph have noncommutative weights (thought of as formal non-commuting variables). In place of an ordinary (commutative) Poisson bracket, we used the notion of a double bracket [VdB08], which is a bilinear operation (denoted $\{\cdot, \cdot\}$) on a (noncommutative) algebra $A$, which takes values in $A \otimes A$ (rather than just $A$). The double bracket is required to satisfy some relations that resemble the axioms of a Poisson algebra.

Our first main result is a direct generalization of the $R$-matrix formula above.

**Theorem 2. [AOS20]** (R-Matrix Formula for Non-commutative Path-Weight Matrices)

Let $B(\lambda)$ be the path-weight matrix of a network with noncommutative edge weights. Then the entries of $B$ satisfy the relations

$$\{ B(\lambda), B(\mu) \}_\tau = [R(\lambda, \mu), B(\lambda)_L \otimes B(\mu)_R]$$

The notations on the right-hand side describe noncommutative versions of the expressions in the classical case. For a matrix $M$ with entries $m_{ij} \in A$, the matrix $M_L$ has entries $m_{ij} \otimes 1 \in A \otimes A$, and similarly $M_R$ has entries $1 \otimes m_{ij}$. The notation $\{\cdot, \cdot\}_\tau$ is the "twisted commutator" which means $[a, b]_\tau = ab - (ba)^\tau$, where $\tau$ is the permutation $(x \otimes y)^\tau = y \otimes x$.

We also proved that the spectral invariants of $B(\lambda)$ Poisson-commute as in the commutative case, but with an extra subtlety. The non-commutative Poisson bracket described above induces a Lie bracket on the vector space $A^\natural := A/[A, A]$. For $a^\natural, b^\natural \in A^\natural$, let $\langle a^\natural, b^\natural \rangle$ denote this induced Lie bracket. Then we have the following statement, giving a kind of "non-commutting integrable system":

**Theorem 3. [AOS20]** (Poisson-Commutativity of Non-commutative Hamiltonians)

Let $\text{tr} \ (B(\lambda)^j) = \sum_i H_{ij} \lambda^i$. Then the elements $H^\natural_{ij} \in A^\natural$ commute under the induced bracket:

$$\langle H_{ij}^\natural, H_{k\ell}^\natural \rangle = 0 \quad \text{(for all } i, j, k, \ell \text{)}$$

Although this result is at first glance formal and algebraic, it has deep implications. There is a result in [CB11] which says that a non-commutative bracket (of the kind described above) on $A^\natural$ induces an ordinary (i.e. a usual commutative) Poisson bracket on the character variety $\text{Rep}_n(A) := \text{Hom}(A, \text{Mat}_n(A))/\text{GL}_n$, where the quotient is by conjugation. So our theorem above says that this non-commutative integrable system gives rise to an infinite family of integrable systems on the spaces $\text{Rep}_n(A)$ for all $n$.

**An Application: The Pentagram Map**

The pentagram map [Sch92] is defined on the space of plane polygons as follows. Given a polygon in the plane, draw the diagonals connecting vertex $i$ to vertex $i + 2$. The intersection points of these diagonals will give the vertices of a new polygon. The map sending the old polygon to the new one is called the pentagram map. It is pictured below:
More generally, we can consider the polygons in the projective plane $\mathbb{P}^2$. A twisted polygon in $\mathbb{P}^2$ is a bi-infinite sequence of points $(p_i)_{i \in \mathbb{Z}}$ in $\mathbb{P}^2$ together with a projective transformation $M \in \text{SL}_3$ (called the monodromy) such that $p_{i+n} = Mp_i$ for each $i$. If we let $\mathcal{P}$ denote the moduli space of twisted polygons in $\mathbb{P}^2$, then the pentagram map makes sense as a map on $\mathcal{P}$. This map was proven to be completely integrable [OST10]. That is, there exists a maximal family of independent functions $f_i$ on $\mathcal{P}$, which are invariant under the pentagram map, and a Poisson structure in which $\{f_i, f_j\} = 0$ for all $i, j$.

Recently, Mari-Beffa and Felipe introduced a generalization of the pentagram map where the twisted polygons are in a Grassmannian (rather than projective space) [FMB19].

As an application of the techniques mentioned above, I interpreted the space of Grassmann polygons as the space of matrix-valued weights on some cylindrical network to obtain some statement of integrability of this system.

**Theorem 4.** [Ove20] (Integrability of Grassmann Pentagram Map)

(a) The space of Grassmann polygons is isomorphic to the space of matrix-valued weights on a cylindrical network.

(b) The induced bracket $\langle -, - \rangle$ on $A^1 = A/[A, A]$ gives an actual Poisson bracket on the space of Grassmann polygons.

(c) Let $H_{ij}$ be as in Theorem 3. They are invariant under the Grassmannian pentagram map, and they Poisson-commute.

**References**


