Noncommutative Poisson Structures from Networks

Nick Ovenhouse
(joint with Semeon Arthamonov and Michael Shapiro)

University of Minnesota

April, 2022
Gone Fishing Poisson Geometry Conference

Noncommutative Poisson Structures

Noncommutative Poisson Structures

For an associative algebra A, let $A^\natural$ be the vector space quotient $A/[A, A]$, called the *cyclic space.*

Noncommutative Poisson Structures

For an associative algebra A, let $A^\natural$ be the vector space quotient $A/[A, A]$, called the *cyclic space.*

For example $(abc)^\natural = (cab)^\natural = (bca)^\natural$.

Noncommutative Poisson Structures

For an associative algebra A, let $A^\natural$ be the vector space quotient $A/[A, A]$, called the *cyclic space.*

For example $(abc)^\natural = (cab)^\natural = (bca)^\natural$.

Definition

A "*noncommutate Poisson bracket*" on A is a Lie bracket on $A^\natural$ such that each $\mathrm{ad}_{a^\natural} = [a^\natural, -]$ comes from a derivation of A.

Noncommutative Poisson Structures

For an associative algebra A, let $A^\natural$ be the vector space quotient $A/[A, A]$, called the *cyclic space*.

For example $(abc)^\natural = (cab)^\natural = (bca)^\natural$.

Definition

A "*noncommutate Poisson bracket*" on A is a Lie bracket on $A^\natural$ such that each $\mathrm{ad}_{a^\natural} = [a^\natural, -]$ comes from a derivation of A.

Theorem [Crawley-Boevey][1]

A noncommutative Poisson bracket on A induces an ordinary commutative Poisson bracket on the moduli space of representations $\mathrm{Rep}_n(A) := \mathrm{Hom}(A, \mathrm{Mat}_n)/\mathrm{GL}_n$:

$$\{\mathrm{tr}(a), \mathrm{tr}(b)\} := \mathrm{tr}[a^\natural, b^\natural]$$

[1] William Crawley-Boevey. "Poisson Structures on Moduli Spaces of Representations". In: *Journal of Algebra* 325.1 (2011), pp. 205–215. DOI: 10.1016/j.jalgebra.2010.09.033

Double Brackets

Definition [Van den Bergh][1]

A *double bracket* on A is a bilinear map $\{\!\{-,-\}\!\} : A \times A \to A \otimes A$ such that

[1] Michel Van Den Bergh. “Double Poisson Algebras”. In: *Transactions of the American Mathematical Society* 360.11 (2008), pp. 5711–5769

Double Brackets

Definition [Van den Bergh][1]

A *double bracket* on A is a bilinear map $\{\!\{-,-\}\!\} : A \times A \to A \otimes A$ such that

- $\{\!\{a, bc\}\!\} = \{\!\{a, b\}\!\} (1 \otimes c) + (b \otimes 1) \{\!\{a, c\}\!\}$

[1] Michel Van Den Bergh. “Double Poisson Algebras”. In: *Transactions of the American Mathematical Society* 360.11 (2008), pp. 5711–5769

Double Brackets

Definition [Van den Bergh][1]

A *double bracket* on A is a bilinear map $\{\!\{-,-\}\!\} : A \times A \to A \otimes A$ such that

- $\{\!\{a, bc\}\!\} = \{\!\{a, b\}\!\}(1 \otimes c) + (b \otimes 1)\{\!\{a, c\}\!\}$
- $\{\!\{b, a\}\!\} = -\{\!\{a, b\}\!\}^{\tau}$, where τ means $(x \otimes y)^{\tau} = y \otimes x$

[1] Michel Van Den Bergh. "Double Poisson Algebras". In: *Transactions of the American Mathematical Society* 360.11 (2008), pp. 5711–5769

Double Brackets

Definition [Van den Bergh][1]

A *double bracket* on A is a bilinear map $\{\!\{-,-\}\!\} : A \times A \to A \otimes A$ such that

- $\{\!\{a, bc\}\!\} = \{\!\{a, b\}\!\} (1 \otimes c) + (b \otimes 1) \{\!\{a, c\}\!\}$
- $\{\!\{b, a\}\!\} = - \{\!\{a, b\}\!\}^{\tau}$, where τ means $(x \otimes y)^{\tau} = y \otimes x$

[1] Michel Van Den Bergh. "Double Poisson Algebras". In: *Transactions of the American Mathematical Society* 360.11 (2008), pp. 5711–5769

Double Brackets

Definition [Van den Bergh][1]

A *double bracket* on A is a bilinear map $\{\!\{-,-\}\!\} : A \times A \to A \otimes A$ such that

- $\{\!\{a, bc\}\!\} = \{\!\{a, b\}\!\}(1 \otimes c) + (b \otimes 1)\{\!\{a, c\}\!\}$
- $\{\!\{b, a\}\!\} = -\{\!\{a, b\}\!\}^{\tau}$, where τ means $(x \otimes y)^{\tau} = y \otimes x$

Note: There is a "double Jacobi identity" and a "double quasi Jacobi identity", but we do not write them here.

[1] Michel Van Den Bergh. "Double Poisson Algebras". In: *Transactions of the American Mathematical Society* 360.11 (2008), pp. 5711–5769

Induced Bracket

Induced Bracket

Compose with the multiplication map $\mu\colon A \otimes A \to A$, and the quotient $A \to A^\natural = A/[A, A]$ to get an operation on $A^\natural$:

$$[a^\natural, b^\natural] := \Big(\mu\left(\{\!\{a, b\}\!\}\right)\Big)^\natural$$

Induced Bracket

Compose with the multiplication map $\mu\colon A \otimes A \to A$, and the quotient $A \to A^\natural = A/[A, A]$ to get an operation on $A^\natural$:

$$[a^\natural, b^\natural] := \Big(\mu\,(\{\!\{a, b\}\!\})\Big)^\natural$$

This means if $\{\!\{a, b\}\!\} = \sum_i x_i \otimes y_i$, then

$$[a^\natural, b^\natural] = \sum_i (x_i y_i)^\natural$$

Induced Bracket

Compose with the multiplication map $\mu\colon A \otimes A \to A$, and the quotient $A \to A^\natural = A/[A, A]$ to get an operation on $A^\natural$:

$$[a^\natural, b^\natural] := \Big(\mu\left(\{\!\{a, b\}\!\}\right)\Big)^\natural$$

This means if $\{\!\{a, b\}\!\} = \sum_i x_i \otimes y_i$, then

$$[a^\natural, b^\natural] = \sum_i (x_i y_i)^\natural$$

If $\{\!\{-, -\}\!\}$ satisfies some version of the Jacobi identity, then this will be a noncommutative Poisson structure, and induce an ordinary (commutative) Poisson structure on the spaces $\mathrm{Rep}_n(A)$.

Poisson Structures on Matrix Spaces

Poisson Structures on Matrix Spaces

Let M be the space of n-by-n matrices, and A its ring of functions.

Poisson Structures on Matrix Spaces

Let M be the space of n-by-n matrices, and A its ring of functions.

Let x_{ij} be the coordinate functions on M.

Poisson Structures on Matrix Spaces

Let M be the space of n-by-n matrices, and A its ring of functions.

Let x_{ij} be the coordinate functions on M.

Define $X \in \mathrm{Mat}_n(A)$ with entries x_{ij}.

Poisson Structures on Matrix Spaces

Let M be the space of n-by-n matrices, and A its ring of functions.

Let x_{ij} be the coordinate functions on M.

Define $X \in \mathrm{Mat}_n(A)$ with entries x_{ij}.

The Poisson matrix of any Poisson bracket on A can be thought of as an element of $M \otimes M$ which we denote "$\{X, X\}$":

$$\{X, X\} := \sum_{ij,k\ell} \{x_{ij}, x_{k\ell}\} e_{ij} \otimes e_{k\ell}$$

Example: 2×2

If $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\{X, X\} = \left(\begin{array}{cc|cc} \{a,a\} & \{a,b\} & \{b,a\} & \{b,b\} \\ \{a,c\} & \{a,d\} & \{b,c\} & \{b,d\} \\ \hline \{c,a\} & \{c,b\} & \{d,a\} & \{d,b\} \\ \{c,c\} & \{c,d\} & \{d,c\} & \{d,d\} \end{array}\right)$$

R-Matrix Brackets

R-Matrix Brackets

Consider the element $R \in M \otimes M$:

$$R = \sum_{i<j} \left(e_{ji} \otimes e_{ij} - e_{ij} \otimes e_{ji} \right)$$

R-Matrix Brackets

Consider the element $R \in M \otimes M$:

$$R = \sum_{i<j} \left(e_{ji} \otimes e_{ij} - e_{ij} \otimes e_{ji} \right)$$

We define a bracket on M by declaring that

$$\{X, X\} = \left[R,\ X \otimes X \right]$$

R-Matrix Brackets

Consider the element $R \in M \otimes M$:

$$R = \sum_{i<j} (e_{ji} \otimes e_{ij} - e_{ij} \otimes e_{ji})$$

We define a bracket on M by declaring that

$$\{X, X\} = \left[R,\ X \otimes X\right]$$

Theorem

The functions $H_k := \mathrm{tr}(X^k)$ Poisson-commute:

$$\{H_i, H_j\} = 0 \quad \text{for all } i, j$$

Example: 2-by-2

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Example: 2-by-2

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\text{Then} \quad R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X \otimes X = \left(\begin{array}{cc|cc} a^2 & ab & ba & b^2 \\ ac & ad & bc & bd \\ \hline ca & cb & da & db \\ c^2 & cd & dc & d^2 \end{array}\right)$$

Example: 2-by-2

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\text{Then} \quad R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X \otimes X = \left(\begin{array}{cc|cc} a^2 & ab & ba & b^2 \\ ac & ad & bc & bd \\ \hline ca & cb & da & db \\ c^2 & cd & dc & d^2 \end{array}\right)$$

Then $\{X, X\}$ is given by

$$\{X, X\} = [R, X \otimes X] = \left(\begin{array}{cc|cc} 0 & -ab & ab & 0 \\ -ac & -2bc & 0 & -bd \\ \hline ac & 0 & 2bc & bd \\ 0 & -cd & cd & 0 \end{array}\right)$$

Example: 2-by-2

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\text{Then} \quad R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X \otimes X = \left(\begin{array}{cc|cc} a^2 & ab & ba & b^2 \\ ac & ad & bc & bd \\ \hline ca & cb & da & db \\ c^2 & cd & dc & d^2 \end{array}\right)$$

Then $\{X, X\}$ is given by

$$\{X, X\} = [R, X \otimes X] = \left(\begin{array}{cc|cc} 0 & -ab & ab & 0 \\ -ac & -2bc & 0 & -bd \\ \hline ac & 0 & 2bc & bd \\ 0 & -cd & cd & 0 \end{array}\right)$$

This means that

$$\{b, a\} = ba \qquad \{c, a\} = ca$$
$$\{d, b\} = db \qquad \{d, c\} = dc$$
$$\{d, a\} = 2bc \qquad \{b, c\} = 0$$

Noncommutative Version

Noncommutative Version

Let $\mathcal{A}$ be the free associative (but not commutative!) algebra generated by x_{ij}.

Noncommutative Version

Let $\mathcal{A}$ be the free associative (but not commutative!) algebra generated by x_{ij}.

Question: Is there a noncommutative Poisson bracket on $\mathcal{A}$ which resembles the R-matrix Poisson bracket?

Some Definitions

Some Definitions

For $x \otimes y \in \mathcal{A} \otimes \mathcal{A}$, let $(x \otimes y)^\tau := y \otimes x$.

Some Definitions

For $x \otimes y \in \mathcal{A} \otimes \mathcal{A}$, let $(x \otimes y)^\tau := y \otimes x$.

For $X, Y \in \mathrm{Mat}_n(\mathcal{A} \otimes \mathcal{A})$, define the *twisted commutator*

$$[X, Y]_\tau := XY - (YX)^\tau \qquad \text{(apply } \tau \text{ element-wise)}$$

Some Definitions

For $x \otimes y \in \mathcal{A} \otimes \mathcal{A}$, let $(x \otimes y)^\tau := y \otimes x$.

For $X, Y \in \mathrm{Mat}_n(\mathcal{A} \otimes \mathcal{A})$, define the *twisted commutator*

$$[X, Y]_\tau := XY - (YX)^\tau \qquad \text{(apply } \tau \text{ element-wise)}$$

For $X \in \mathrm{Mat}_n(\mathcal{A})$, define "left" and "right" matrices over $\mathcal{A} \otimes \mathcal{A}$:

$$X_L = (x_{ij} \otimes 1)_{i,j=1}^n \qquad \text{and} \qquad X_R = (1 \otimes x_{ij})_{i,j=1}^n$$

Noncommutative R-Matrix Bracket

Noncommutative R-Matrix Bracket

If $X = (x_{ij})_{i,j=1}^{n}$, then a double bracket on $\mathcal{A}$ is described by $\{\!\{X, X\}\!\} \in \mathrm{Mat}_n(\mathcal{A} \otimes \mathcal{A}) \otimes \mathrm{Mat}_n(\mathcal{A} \otimes \mathcal{A})$:

$$\{\!\{X, X\}\!\} = \sum_{ij,k\ell} \{\!\{x_{ij}, x_{k\ell}\}\!\}\, e_{ij} \otimes e_{k\ell}$$

Noncommutative *R*-Matrix Bracket

If $X = (x_{ij})_{i,j=1}^{n}$, then a double bracket on $\mathcal{A}$ is described by $\{\!\{X, X\}\!\} \in \mathrm{Mat}_n(\mathcal{A} \otimes \mathcal{A}) \otimes \mathrm{Mat}_n(\mathcal{A} \otimes \mathcal{A})$:

$$\{\!\{X, X\}\!\} = \sum_{ij,k\ell} \{\!\{x_{ij}, x_{k\ell}\}\!\}\, e_{ij} \otimes e_{k\ell}$$

Define a double bracket on $\mathcal{A}$ by

$$\{\!\{X, X\}\!\} = \Big[R,\ X_L \otimes X_R\Big]_{\tau}$$

Noncommutative R-Matrix Bracket

If $X = (x_{ij})_{i,j=1}^{n}$, then a double bracket on $\mathcal{A}$ is described by $\{\!\{X, X\}\!\} \in \mathrm{Mat}_n(\mathcal{A} \otimes \mathcal{A}) \otimes \mathrm{Mat}_n(\mathcal{A} \otimes \mathcal{A})$:

$$\{\!\{X, X\}\!\} = \sum_{ij,k\ell} \{\!\{x_{ij}, x_{k\ell}\}\!\}\, e_{ij} \otimes e_{k\ell}$$

Define a double bracket on $\mathcal{A}$ by

$$\{\!\{X, X\}\!\} = \Big[R,\ X_L \otimes X_R\Big]_{\tau}$$

Theorem [Arthamonov, N.O., Shapiro][1]

This double bracket satisfies the double quasi-Jacobi identity, and so it gives a noncommutative Poisson bracket on $\mathcal{A}$.

[1] S Arthamonov, N Ovenhouse, and M Shapiro. "Noncommutative Networks on a Cylinder". In: *arXiv preprint arXiv:2008.02889* (2020)

Example: 2-by-2

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Example: 2-by-2

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then R is the same as the commutative case, and

$$X_L \otimes X_R = \left(\begin{array}{cc|cc} a \otimes a & a \otimes b & b \otimes a & b \otimes b \\ a \otimes c & a \otimes d & b \otimes c & b \otimes d \\ \hline c \otimes a & c \otimes b & d \otimes a & d \otimes b \\ c \otimes c & c \otimes d & d \otimes c & d \otimes d \end{array}\right)$$

Example: 2-by-2

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then R is the same as the commutative case, and

$$X_L \otimes X_R = \left(\begin{array}{cc|cc} a \otimes a & a \otimes b & b \otimes a & b \otimes b \\ a \otimes c & a \otimes d & b \otimes c & b \otimes d \\ \hline c \otimes a & c \otimes b & d \otimes a & d \otimes b \\ c \otimes c & c \otimes d & d \otimes c & d \otimes d \end{array}\right)$$

Then $\{\!\{X, X\}\!\}$ is given by

$$\{\!\{X, X\}\!\} = [R,\, X_L \otimes X_R]_\tau = \left(\begin{array}{cc|cc} 0 & -a \otimes b & b \otimes a & 0 \\ -c \otimes a & -2c \otimes b & 0 & -d \otimes b \\ \hline a \otimes c & 0 & 2b \otimes c & b \otimes d \\ 0 & -c \otimes d & d \otimes c & 0 \end{array}\right)$$

Example: 2-by-2

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then R is the same as the commutative case, and

$$X_L \otimes X_R = \left(\begin{array}{cc|cc} a\otimes a & a\otimes b & b\otimes a & b\otimes b \\ a\otimes c & a\otimes d & b\otimes c & b\otimes d \\ \hline c\otimes a & c\otimes b & d\otimes a & d\otimes b \\ c\otimes c & c\otimes d & d\otimes c & d\otimes d \end{array}\right)$$

Then $\{\!\{X, X\}\!\}$ is given by

$$\{\!\{X, X\}\!\} = [R,\ X_L \otimes X_R]_\tau = \left(\begin{array}{cc|cc} 0 & -a\otimes b & b\otimes a & 0 \\ -c\otimes a & -2c\otimes b & 0 & -d\otimes b \\ \hline a\otimes c & 0 & 2b\otimes c & b\otimes d \\ 0 & -c\otimes d & d\otimes c & 0 \end{array}\right)$$

This means that

$$\{\!\{b, a\}\!\} = b\otimes a \qquad \{\!\{c, a\}\!\} = a\otimes c$$
$$\{\!\{d, b\}\!\} = b\otimes d \qquad \{\!\{d, c\}\!\} = d\otimes c$$
$$\{\!\{d, a\}\!\} = 2b\otimes c \qquad \{\!\{b, c\}\!\} = 0$$

Commuting Hamiltonians

Commuting Hamiltonians

Theorem [Arthamonov, N.O., Shapiro][1]

Let $H_k = \mathrm{tr}(X^k) \in \mathcal{A}$. Then the H_k's commute under the bracket $[-,-]$ on $\mathcal{A}^\natural$:

$$\left[H_i^\natural,\, H_j^\natural\right] = 0$$

[1] S Arthamonov, N Ovenhouse, and M Shapiro. "Noncommutative Networks on a Cylinder". In: *arXiv preprint arXiv:2008.02889* (2020)

Commuting Hamiltonians

Theorem [Arthamonov, N.O., Shapiro][1]

Let $H_k = \mathrm{tr}(X^k) \in \mathcal{A}$. Then the H_k's commute under the bracket $[-,-]$ on $\mathcal{A}^\natural$:

$$\left[H_i^\natural,\, H_j^\natural\right] = 0$$

Example: For 2×2 case, $H_1 = a + d$ and $H_2 = a^2 + bc + cb + d^2$, and

[1] S Arthamonov, N Ovenhouse, and M Shapiro. "Noncommutative Networks on a Cylinder". In: *arXiv preprint arXiv:2008.02889* (2020)

Commuting Hamiltonians

Theorem [Arthamonov, N.O., Shapiro][1]

Let $H_k = \mathrm{tr}(X^k) \in \mathcal{A}$. Then the H_k's commute under the bracket $[-,-]$ on $\mathcal{A}^\natural$:

$$\left[H_i^\natural,\, H_j^\natural\right] = 0$$

Example: For 2×2 case, $H_1 = a + d$ and $H_2 = a^2 + bc + cb + d^2$, and

$$\begin{aligned} \{\!\{H_1, H_2\}\!\} = {} & 2ab \otimes c + 2b \otimes ca \\ & - a \otimes bc - bc \otimes a - c \otimes ab - ca \otimes b \\ & + b \otimes dc + cb \otimes d + bd \otimes c + d \otimes cb \\ & - 2c \otimes bd - 2dc \otimes b \end{aligned}$$

[1] S Arthamonov, N Ovenhouse, and M Shapiro. "Noncommutative Networks on a Cylinder". In: *arXiv preprint arXiv:2008.02889* (2020)

Commuting Hamiltonians

Theorem [Arthamonov, N.O., Shapiro][1]

Let $H_k = \mathrm{tr}(X^k) \in \mathcal{A}$. Then the H_k's commute under the bracket $[-,-]$ on $\mathcal{A}^\natural$:

$$\left[H_i^\natural,\, H_j^\natural\right] = 0$$

Example: For 2×2 case, $H_1 = a + d$ and $H_2 = a^2 + bc + cb + d^2$, and

$$\begin{aligned}\{\!\{H_1, H_2\}\!\} = {} & 2ab \otimes c + 2b \otimes ca \\ & - a \otimes bc - bc \otimes a - c \otimes ab - ca \otimes b \\ & + b \otimes dc + cb \otimes d + bd \otimes c + d \otimes cb \\ & - 2c \otimes bd - 2dc \otimes b\end{aligned}$$

$$\begin{aligned}\mu\Big(\{\!\{H_1, H_2\}\!\}\Big) = {} & abc + bca - 2cab \\ & + 2bdc - cbd - dcb\end{aligned}$$

[1] S Arthamonov, N Ovenhouse, and M Shapiro. "Noncommutative Networks on a Cylinder". In: *arXiv preprint arXiv:2008.02889* (2020)

Application: Planar Networks

Application: Planar Networks

Let Γ be a weighted planar directed graph in a disc (drawn as a rectangle) with:

Application: Planar Networks

Let Γ be a weighted planar directed graph in a disc (drawn as a rectangle) with:

- n univalent sources on the left boundary

Application: Planar Networks

Let Γ be a weighted planar directed graph in a disc (drawn as a rectangle) with:

- n univalent sources on the left boundary
- n univalent sinks on the right boundary

Application: Planar Networks

Let Γ be a weighted planar directed graph in a disc (drawn as a rectangle) with:

- n univalent sources on the left boundary
- n univalent sinks on the right boundary
- all internal vertices trivalent, neither sources nor sinks

Application: Planar Networks

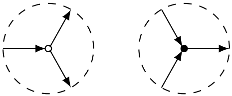
Let Γ be a weighted planar directed graph in a disc (drawn as a rectangle) with:

- n univalent sources on the left boundary
- n univalent sinks on the right boundary
- all internal vertices trivalent, neither sources nor sinks
- a weight function $\{\text{edges}\} \to \mathbb{R}$

Application: Planar Networks

Let Γ be a weighted planar directed graph in a disc (drawn as a rectangle) with:

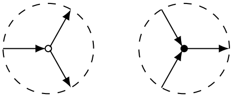
- n univalent sources on the left boundary
- n univalent sinks on the right boundary
- all internal vertices trivalent, neither sources nor sinks
- a weight function $\{\text{edges}\} \to \mathbb{R}$



Application: Planar Networks

Let Γ be a weighted planar directed graph in a disc (drawn as a rectangle) with:

- n univalent sources on the left boundary
- n univalent sinks on the right boundary
- all internal vertices trivalent, neither sources nor sinks
- a weight function $\{\text{edges}\} \to \mathbb{R}$

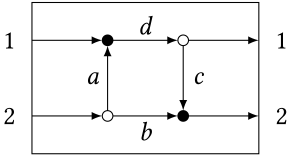


Definition

The *boundary measurement matrix* is the matrix $B = (b_{ij})$ where

$$b_{ij} = \sum_{p:\ i \to j} \mathrm{wt}(p)$$

Example



"*boundary measurement matrix*"

$$B = \begin{pmatrix} d & dc \\ ad & b + adc \end{pmatrix}$$

Poisson Structure

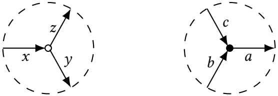
Poisson Structure

Let E be the space of weights on Γ.

Poisson Structure

Let E be the space of weights on Γ.

Define a Poisson structure on E: at each vertex, define

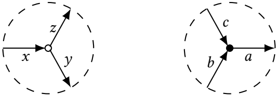


$$\{y, z\} = yz \qquad \text{and} \qquad \{b, c\} = bc$$

Poisson Structure

Let E be the space of weights on Γ.

Define a Poisson structure on E: at each vertex, define



$$\{y, z\} = yz \qquad \text{and} \qquad \{b, c\} = bc$$

Theorem [Gekhtman, Shapiro, Vainshtein][1]

The boundary measurement map $E \to \mathrm{Mat}_n(\mathbb{R})$ is a Poisson map (w.r.t the R-matrix bracket on Mat_n). In other words,

$$\{B,\ B\} = \left[R,\ B \otimes B\right]$$

[1] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein. “Poisson Geometry of Directed Networks in a Disk”. In: *Selecta Mathematica, New Series* 15.1 (2009), pp. 61–103

Noncommutative Networks

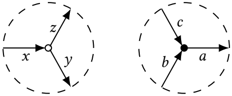
Noncommutative Networks

For a network Γ, let $\mathcal{A}$ be the free associative algebra generated by variables associated to the edges of Γ.

Noncommutative Networks

For a network Γ, let $\mathcal{A}$ be the free associative algebra generated by variables associated to the edges of Γ.

Define a double bracket:



$$\{\!\{y, z\}\!\} = y \otimes z \qquad \text{and} \qquad \{\!\{b, c\}\!\} = c \otimes b$$

Noncommutative Networks

Noncommutative Networks

Let B be the boundary measurement matrix of a network (with entries in $\mathcal{A}$).

Noncommutative Networks

Let B be the boundary measurement matrix of a network (with entries in $\mathcal{A}$).

Theorem [Arthamonov, N.O., Shapiro][1]

The entries of B satisfy the R-matrix double bracket relations:

$$\{\!\{B, B\}\!\} = [R, B_L \otimes B_R]_\tau$$

[1]S Arthamonov, N Ovenhouse, and M Shapiro. "Noncommutative Networks on a Cylinder". In: *arXiv preprint arXiv:2008.02889* (2020)

Noncommutative Networks

Let B be the boundary measurement matrix of a network (with entries in $\mathcal{A}$).

Theorem [Arthamonov, N.O., Shapiro][1]

The entries of B satisfy the R-matrix double bracket relations:

$$\{\!\{B,\, B\}\!\} = [R,\, B_L \otimes B_R]_\tau$$

Glue the left and right sides of the network together to get a cylinder. Then the Hamiltonians $H_k = \mathrm{tr}(B^k)$ are sums of loops which go around the cylinder k times.

[1] S Arthamonov, N Ovenhouse, and M Shapiro. "Noncommutative Networks on a Cylinder". In: *arXiv preprint arXiv:2008.02889* (2020)

Thank You!