# $q$-Rationals and Finite Schubert Varieties

Nick Ovenhouse

University of Minnesota

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# $q$-Integers

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and the "$q$-binomial coefficients"

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$$

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- If $f(n)$ counts some set $A_n$, then the $q$-analogue of $f(n)$ is a generating function for some statistic on $A_n$.

- The $q$-analogue of $f(n)$ counts the size of some algebraic variety over $\mathbb{F}_q$.

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where $\mathrm{Fl}(n)$ is the set of complete flags in $(\mathbb{F}_q)^n$.

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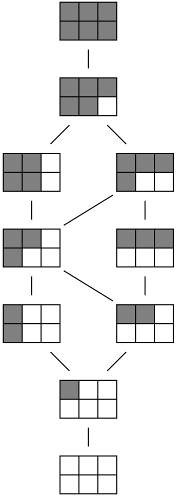
$$\binom{n}{k}_q = \sum_{\mu \leq \lambda} q^{|\mu|}$$

where $\lambda$ is the partition whose Young diagram is a $k \times (n-k)$ rectangle.



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Also, if $P_{n,k}$ is the set of north-east lattice paths:

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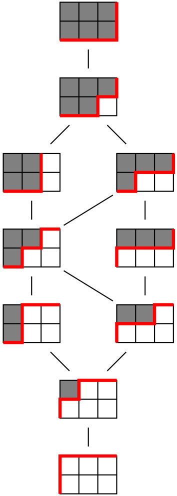

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Also, for the $q$-binomial coefficients, we have

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**But this is not interesting!**

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Given a rational number $\frac{r}{s} > 1$, there is a unique continued fraction expansion of the form

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**Example:**

$$\frac{10}{7} = [1, 2, 2, 1]$$

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$$\left[\frac{r}{s}\right]_q := [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{[a_3]_q + \cfrac{q^{a_3}}{[a_4]_{q^{-1}} + \cfrac{q^{-a_4}}{\ddots + \cfrac{q^{a_{2m-1}}}{[a_{2m}]_{q^{-1}}}}}}}$$

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The group $\mathrm{PSL}_2(\mathbb{Z})$ acts transitively on $\mathbb{Q} \cup \{\infty\}$ by

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**Example:** We can send 1 to $\frac{10}{7}$ by $AB^2A^2$:

$$1 \xrightarrow{A^2} 3 \xrightarrow{B} \frac{3}{4} \xrightarrow{B} \frac{3}{7} \xrightarrow{A} \frac{10}{7}$$

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Define $q$-versions of the matrices $A$ and $B$:

$$A_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}$$

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## Theorem [Morier-Genoud, Leclere][1]

The $\mathrm{PSL}_2(\mathbb{Z})$ action commutes with $q$-deformation. That is, if $x \in \mathbb{Q}$, then

$$[M \cdot x]_q = M_q \cdot [x]_q$$

[1] Ludivine Leclere and Sophie Morier-Genoud. "q-Deformations in the modular group and of the real quadratic irrational numbers". In: *Advances in Applied Mathematics* 130 (2021), p. 102223

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Then we get

$$1 \xrightarrow{A_q^2} 1 + q + q^2 \xrightarrow{B_q^2} \frac{q^2+q^3+q^4}{1+q+2q^2+2q^3+q^4} \xrightarrow{A_q} \frac{1+q+2q^2+3q^3+2q^4+q^5}{1+q+2q^2+2q^3+q^4}$$

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If $\left[\frac{r}{s}\right]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$, then:

**(a)** What are the combinatorial interpretations of $\mathcal{R}$ and $\mathcal{S}$?

**(b)** What are the geometric interpretations of $\mathcal{R}$ and $\mathcal{S}$?

# Snake Graphs

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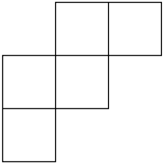
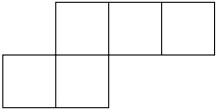
A *snake graph* is a graph made of square tiles, each either above or to the right of the previous.





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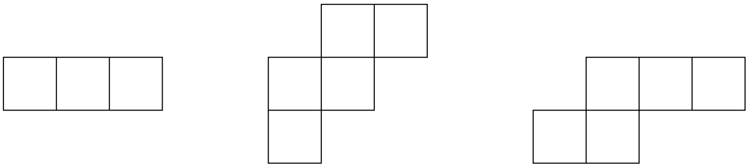

Each snake graph $G$ has a word $W(G)$, in the alphabet $\{U, R\}$, specifying when it goes *up* or *right*.

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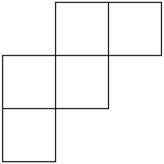

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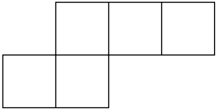

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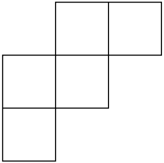
$W(G) = RR$

$G = G(4/1)$

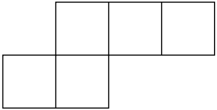
$4/1 = [3, 1]$



$W(G) = URUR$

$G = G(13/8)$

$13/8 = [1, 1, 1, 1, 1, 1]$



$W(G) = RURR$

$G = G(11/4)$

$11/4 = [2, 1, 2, 1]$

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Let $\frac{r}{s} = [a_1, a_2, \ldots, a_{2m}]$, and $\frac{r'}{s'} = [a_2, a_3, \ldots, a_{2m}]$.

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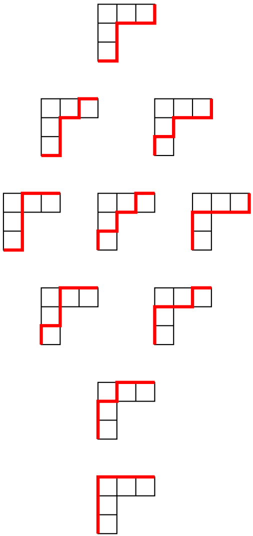

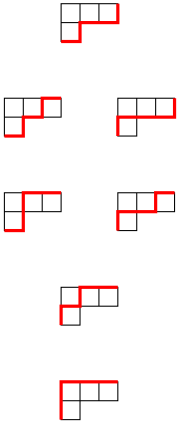

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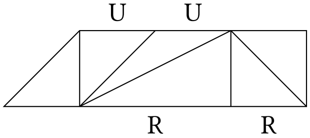

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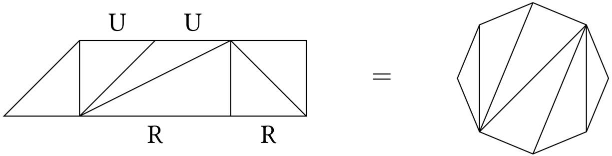

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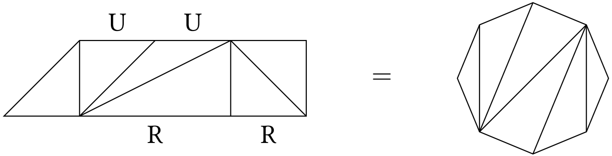


This defines a cluster algebra and an initial cluster.

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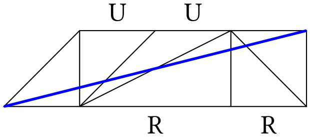
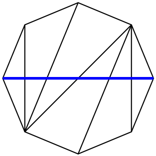
Given a rational number $\frac{r}{s}$, one can associate a triangulated polygon. Reading $W(r/s)$, put a “downward” triangle for “$R$” and an “upward” triangle for “$U$”, and finnally put one extra triangle at the beginning and end.

**Example:** $\frac{10}{7} = [1, 2, 2, 1]$ has word $W(10/7) = UURR$.




$=$



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The “longest” arc corresponds to some cluster variable.

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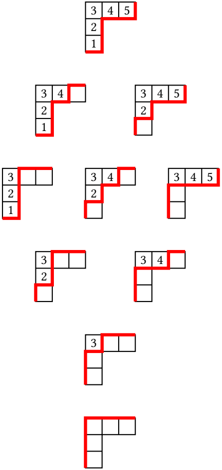

$$\begin{aligned} F(y_1,\ldots,y_5) = 1 &+ y_3 + y_2y_3 + y_3y_4 \\ &+ y_1y_2y_3 + y_2y_3y_4 + y_3y_4y_5 \\ &+ y_1y_2y_3y_4 + y_2y_3y_4y_5 \\ &+ y_1y_2y_3y_4y_5 \end{aligned}$$

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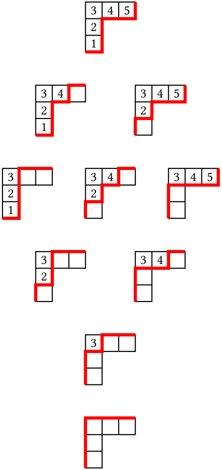


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$$\mathcal{R}(q) = F(q, q, \ldots, q)$$

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For a partition $\lambda$ which fits into a $k \times (n-k)$ rectangle, the "*open Schubert cell*" $\Omega_\lambda^\circ \subset \mathrm{Gr}_k(n)$ is the set of all subspaces represented by matrices whose echelon form is "shaped like $\lambda$".

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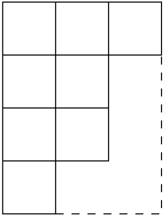

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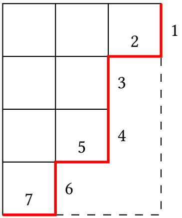

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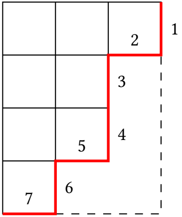


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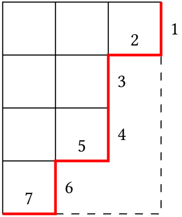


$$\begin{pmatrix} 1 & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}$$

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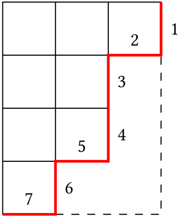


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This explains the equality between the combinatorial and geometric expressions for the $q$-binomial coefficients $\binom{n}{k}_q$:

$$|\mathrm{Gr}_k(n)| = \sum_\lambda |\Omega_\lambda^\circ| = \sum_\lambda q^{|\lambda|}$$

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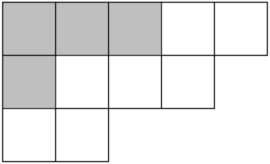

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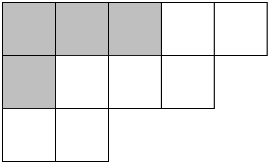

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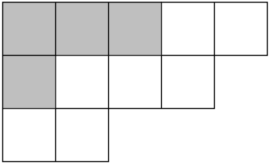

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If $G = G(r/s)$, then write $\lambda(r/s)$ and $\mu(r/s)$.

# Geometric Interpretation of $q$-Rationals

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## Theorem [O][1]

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$$n = \sum_{i=1}^{2m} a_i \quad \text{and} \quad k = \sum_{i=1}^{m} a_{2i}$$

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Then the numerator $\mathcal{R}(q)$ of $\left[\frac{r}{s}\right]_q$ counts the number of points in a union of Schubert cells in $\mathrm{Gr}_k(n)$:

$$q^{|\mu|} \cdot \mathcal{R}(q) = \left| \bigcup_{\mu \leq \nu \leq \lambda} \Omega_\nu^\circ \right|$$

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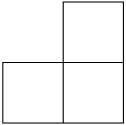

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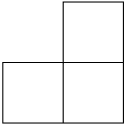

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Then $q \cdot \mathcal{R}(q) = q(1 + 2q + q^2 + q^3) = q + 2q^2 + q^3 + q^4$ counts the number of 2-dimensional subspaces of $(\mathbb{F}_q)^4$ of the forms:

$$\begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix},$$

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# Thank You!