Higher Dimer Covers and Continued Fractions

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Dimers: Combinatorics, Representation Theory and Physics
A *snake graph* is a planar graph, built out of square tiles, where each is attached to either to the right or top edge of the previous.

**Examples:**

\[ W(G) = RR \]

\[ W(G) = URRU \]

\[ W(G) = RURUR \]

A snake graph \( G \) is naturally described by a word \( W(G) \) in the alphabet \( \{ R, U \} \) ("right" and "up").
Duality

Let \( x \mapsto \bar{x} \) be the involution on the set \( \{ R, U \} \).
If \( W = w_1 w_2 \cdots w_n \) is a word, define the *dual word* \( W^* \) to be

\[
W^* = \bar{w}_1 \bar{w}_2 \bar{w}_3 w_4 \cdots \bar{w}_{2k-1} w_{2k} \cdots
\]

For a continued fraction \([a_1, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} \), we associate a snake graph \( G = G[a_1, \ldots, a_n] \), defined so that \( W(G)^* = R^{a_1-1} U a_2 R a_3 U a_4 \cdots R a_{n-1} U a_n - 1 \).

**Example:** \( \frac{103}{30} = [3, 2, 3, 4] \)

\[
W^* = RRUURRRUUU
\]
\[
W = URRUURUURU
\]
Theorem [Çanakçı, Shiffler]

If \( \frac{r}{s} = [a_1, a_2, \ldots, a_n] \), then the number of dimer covers of \( G[a_1, \ldots, a_n] \) is \( r \), and the number of dimer covers of \( G[a_2, \ldots, a_n] \) is \( s \).

Example: \( \frac{5}{2} = [2, 2] \)

\[ G[2, 2] : \]

\[ G[2] : \]
Higher Dimer Covers

For a graph $G$, an $m$-dimer cover is a multiset of edges so that each vertex is incident to $m$ edges.

- When $m = 1$, this is an ordinary dimer cover (perfect matching)
- When $m = 2$, this is called double dimer cover

Examples:

$m = 2$

$m = 3$

Question: How many $m$-dimer covers on a snake graph $G[a_1, \ldots, a_n]$?
Cluster algebras coming from triangulated surfaces are algebras of functions on the *decorated Teichmüller space*.

In 2019, Penner and Zeitlin defined a supermanifold generalizing the decorated Teichmüller space, by adding anti-commuting variables corresponding to triangles. They described the action of a flip of a triangulation.

**Theorem [Musiker, O, Zhang 2021-2022]**

The super cluster variables satisfy the Laurent phenomenon, and the terms in these Laurent polynomials are indexed by double dimer covers of snake graphs.
Matrices for Continued Fractions

For a positive integer $a \in \mathbb{Z}$, define the matrix

$$\Lambda(a) := \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$$

For a finite continued fraction

$$[a_1, a_2, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\vdots + \cfrac{1}{a_n}}}}$$

if $[a_1, \ldots, a_n] = \frac{r}{s}$, and $[a_1, \ldots, a_{n-1}] = \frac{r'}{s'}$, then

$$\Lambda(a_1)\Lambda(a_2)\cdots\Lambda(a_n) = \begin{pmatrix} r & r' \\ s & s' \end{pmatrix}$$
For $a \in \mathbb{N}$, define the $(m+1) \times (m+1)$ matrix

\[
\Lambda(a) = \begin{pmatrix}
\binom{a}{m} & \binom{a}{m-1} & \cdots & \binom{a}{2} & a & 1 \\
\binom{a}{m-1} & \binom{a}{m-2} & \cdots & a & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\binom{a}{2} & a & 1 & 0 & \cdots & 0 \\
a & 1 & 0 & \cdots & \cdots & 0 \\
1 & 0 & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]

where $\binom{n}{k} = \binom{n+k-1}{k}$.

**Theorem [Musiker, O, Schiffler, Zhang]**

Let $M = \Lambda(a_1) \Lambda(a_2) \cdots \Lambda(a_n)$. Then the upper-left entry of $M$ is the number of $m$-dimer covers of $G[a_1, \ldots, a_n]$. 
For $x = [a_1, \ldots, a_n]$, let $M = \Lambda(a_1) \cdots \Lambda(a_n)$.

Define $r_{i,m}(x) := \frac{M_{m+1-i,1}}{M_{m+1,1}}$.

In other words,

$$
\frac{1}{M_{m+1,1}} M = \left( \begin{array}{ccc}
  r_{m,m}(x) & * & \cdots & * \\
  r_{m-1,m}(x) & * & \cdots & * \\
  \vdots & \ddots & \ddots & \vdots \\
  r_{1,m}(x) & * & \cdots & * \\
  1 & * & \cdots & * 
\end{array} \right)
$$
Recursive Definition

**Base Cases:** Define \( r_{0,m}([a_1, \ldots, a_n]) = 1 \) and \( r_{i,m}([a_1]) = \binom{a}{i} \).

**Otherwise:**

\[
r_{i,m}([a_1, \ldots, a_n]) = \sum_{k=0}^{i} \binom{a_1}{k} \frac{r_{m-i+k,m}([a_2, \ldots, a_n])}{r_{m,m}([a_2, \ldots, a_n])}
\]

For \( m = 1 \), we have the usual recurrence for continued fractions:

\[
r_1([a_1, \ldots, a_n]) = a_1 + \frac{1}{r_1([a_2, \ldots, a_n])}
\]

For \( m = 2 \), this is

\[
r_{1,2}([a_1, \ldots, a_n]) = a_1 + \frac{r_{1,2}([a_2, \ldots, a_n])}{r_{2,2}([a_2, \ldots, a_n])}
\]

\[
r_{2,2}([a_1, \ldots, a_n]) = \binom{a_1}{2} + a_1 \frac{r_{1,2}([a_2, \ldots, a_n])}{r_{2,2}([a_2, \ldots, a_n])} + \frac{1}{r_{2,2}([a_2, \ldots, a_n])}
\]
Theorem [Musiker, O, Schiffler, Zhang]

Let $x = [a_1, \ldots, a_n]$. Then

$$r_{m,m}(x) = \frac{\text{# of } m\text{-dimer covers of } G[a_1, \ldots, a_n]}{\text{# of } m\text{-dimer covers of } G[a_2, \ldots, a_n]}$$
Example \((m = 2)\)

\[
\frac{5}{2} = [2, 2], \quad \Lambda(2) \Lambda(2) = \begin{pmatrix} 14 & 8 & 3 \\ 8 & 5 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \quad r_{2,2}(5/2) = \frac{14}{3}
\]
Theorem [Musiker, O, Schiffler, Zhang]

Let $x = [a_1, a_2, \ldots]$ be an irrational number, and let $x_n = [a_1, \ldots, a_n]$ be its continuants. For fixed $i$ and $m$, the sequence $r_{i,m}(x_n)$ converges.

Theorem [Musiker, O, Schiffler, Zhang]

If $x$ is a quadratic irrational (i.e. its continued fraction is eventually periodic), then $r_{i,m}(x)$ is an algebraic number of degree (at most) $m + 1$. 
Let $G_n$ be the straight snake graph with $n - 1$ squares. The number of dimer covers of $G_n$ are the Fibonacci numbers, and
\[
\lim_{n \to \infty} \frac{f_n}{f_{n-1}} = \varphi = \frac{1+\sqrt{5}}{2}.
\]

Let $\rho_m$ be the length of the longest diagonal in a regular $(2m + 3)$-gon with side length 1. Then
\[
r_{m,m}(\varphi) = \lim_{n \to \infty} \frac{\# \text{ of } m\text{-dimer covers of } G_n}{\# \text{ of } m\text{-dimer covers of } G_{n-1}} = \rho_m
\]

For $m = 2$, the number of double-dimer covers of $G_n$ are 3, 6, 14, 31, \ldots, and the consecutive ratios converge to $\rho_2 = 4 \cos^2(\pi/7) \! - \! 1 \approx 2.247$, whose minimal polynomial is $x^3 - 2x^2 - x + 1$. 
Relation to Plane Partitions

There is a bijection between the set $\Omega_m(G)$ of $m$-dimer covers on a snake graph $G$, and reverse plane partitions on the dual snake graph $G^*$, with all parts at most $m$.

Stanley gave formulas for generating functions of reverse plane partitions.

Using Stanley’s formulas, and considering $G^*$ as a poset (and $\mathcal{L}(G^*)$ the set of linear extensions), we have:

$$\sum_{m \geq 0} |\Omega_m(G)| x^m = \frac{1}{(1 - x)^N} \sum_{\sigma \in \mathcal{L}(G^*)} x^{\text{des} (\sigma)}$$
The set $\Omega_m(G)$ has a natural partial order extending the one for ordinary dimer covers.

Replacing $|\Omega_m(G)|$ in the previous expression with the rank polynomial of $\Omega_m(G)$, we get:

$$\sum_{m \geq 0} \sum_{D \in \Omega_m(G)} q^{|D|} x^m = \frac{1}{(x; q)_N} \sum_{\sigma \in \mathcal{L}(G^*)} q^{\text{maj}(\sigma)} x^{\text{des}(\sigma)}$$

where $(x; q)_N = (1 - x)(1 - qx)(1 - q^2x) \cdots (1 - q^{N-1}x)$. 

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Example

For straight snake graphs, there is only one linear extension, and so:

$$\sum_{m \geq 0} \sum_{D \in \Omega_m(G)} q^{|D|} x^m = \frac{1}{(x; q)_N}$$

By the "$q$-binomial theorem", this means

$$\sum_{D \in \Omega_m(G)} q^{|D|} = \binom{N + m - 1}{m}_q$$
Thank You!