Higher Dimer Covers and Continued Fractions

Nick Ovenhouse

(joint with Gregg Musiker, Ralf Schiffler, and Sylvester Zhang)

Yale University

August, 2023

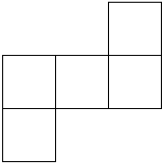
Dimers: Combinatorics, Representation Theory and Physics

Snake Graphs

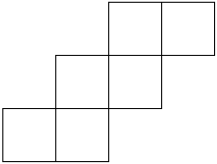
A *snake graph* is a planar graph, built out of square tiles, where each is attached to either to the right or top edge of the previous.

Examples:

$W(G) = RR$



$W(G) = URRU$



$W(G) = RURUR$

A snake graph G is naturally described by a word $W(G)$ in the alphabet $\{R, U\}$ ("*right*" and "*up*").

Duality

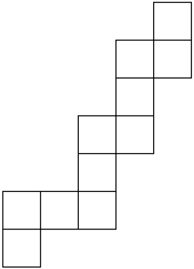
Let $x \mapsto \overline{x}$ be the involution on the set $\{R, U\}$.
If $W = w_1 w_2 \cdots w_n$ is a word, define the *dual word* W^* to be

$$W^* = \overline{w}_1 w_2 \overline{w}_3 w_4 \cdots \overline{w}_{2k-1} w_{2k} \cdots$$

For a continued fraction $[a_1, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$, we associate a snake graph $G = G[a_1, \ldots, a_n]$, defined so that $W(G)^* = R^{a_1 - 1} U^{a_2} R^{a_3} U^{a_4} \cdots R^{a_{n-1}} U^{a_n - 1}$.

Example: $\frac{103}{30} = [3, 2, 3, 4]$

$$W^* = RRUURRRUUU$$
$$W = URRUURUURU$$



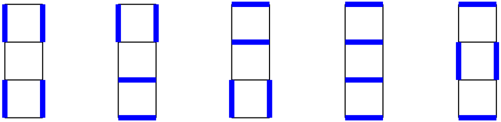
Dimer Cover Enumeration Theorem

Theorem [Çanakçı, Shiffler]

If $\frac{r}{s} = [a_1, a_2, \ldots, a_n]$, then the number of dimer covers of $G[a_1, \ldots, a_n]$ is r, and the number of dimer covers of $G[a_2, \ldots, a_n]$ is s.

Example: $\frac{5}{2} = [2, 2]$

$G[2, 2]$:



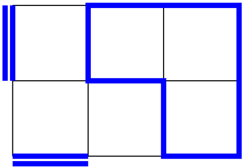
$G[2]$:

For a graph G, an *m-dimer cover* is a multiset of edges so that each vertex is incident to m edges.

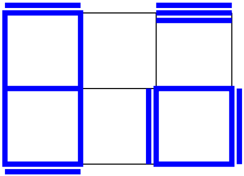
- When $m = 1$, this is an ordinary dimer cover (perfect matching)
- When $m = 2$, this is called *double dimer cover*

Examples:

$m = 2$



$m = 3$



Question: How many m-dimer covers on a snake graph $G[a_1, \ldots, a_n]$?

Why Count m-Dimer Covers?

Cluster algebras coming from triangulated surfaces are algebras of functions on the *decorated Teichmüller space.*

In 2019, Penner and Zeitlin defined a supermanifold generalizing the decorated Teichmüller space, by adding anti-commuting variables corresponding to triangles. They described the action of a flip of a triangulation.

Theorem [Musiker, O, Zhang 2021-2022]

The super cluster variables satisfy the Laurent phenomenon, and the terms in these Laurent polynomials are indexed by double dimer covers of snake graphs.

Matrices for Continued Fractions

For a positive integer $a \in \mathbb{Z}$, define the matrix

$$\Lambda(a) := \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$$

For a finite continued fraction

$$[a_1, a_2, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

if $[a_1, \ldots, a_n] = \frac{r}{s}$, and $[a_1, \ldots, a_{n-1}] = \frac{r'}{s'}$, then

$$\Lambda(a_1)\Lambda(a_2)\cdots\Lambda(a_n) = \begin{pmatrix} r & r' \\ s & s' \end{pmatrix}$$

Bigger Matrices

For $a \in \mathbb{N}$, define the $(m+1) \times (m+1)$ matrix

$$\Lambda(a) = \begin{pmatrix} \left(\!\!\binom{a}{m}\!\!\right) & \left(\!\!\binom{a}{m-1}\!\!\right) & \cdots & \left(\!\!\binom{a}{2}\!\!\right) & a & 1 \\ \left(\!\!\binom{a}{m-1}\!\!\right) & \left(\!\!\binom{a}{m-2}\!\!\right) & \cdots & a & 1 & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\ \left(\!\!\binom{a}{2}\!\!\right) & a & 1 & 0 & \cdots & 0 \\ a & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

where $\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$.

Theorem[Musiker, O, Schiffler, Zhang]

Let $M = \Lambda(a_1)\Lambda(a_2)\cdots\Lambda(a_n)$. Then the upper-left entry of M is the number of m-dimer covers of $G[a_1, \ldots, a_n]$.

Higher Continued Fractions

For $x = [a_1, \ldots, a_n]$, let $M = \Lambda(a_1) \cdots \Lambda(a_n)$.

Define $r_{i,m}(x) := \frac{M_{m+1-i,1}}{M_{m+1,1}}$.
In other words,

$$\frac{1}{M_{m+1,1}} M = \begin{pmatrix} r_{m,m}(x) & * & \cdots & * \\ r_{m-1,m}(x) & * & \cdots & * \\ \vdots & \ddots & \cdots & \vdots \\ r_{1,m}(x) & * & \cdots & * \\ 1 & * & \cdots & * \end{pmatrix}$$

Recursive Definition

Base Cases: Define $r_{0,m}([a_1, \ldots, a_n]) = 1$ and $r_{i,m}([a_1]) = \left(\!\binom{a}{i}\!\right)$.

Otherwise: $r_{i,m}([a_1, \ldots, a_n]) = \sum_{k=0}^{i} \left(\!\!\binom{a_1}{k}\!\!\right) \frac{r_{m-i+k,m}([a_2, \ldots, a_n])}{r_{m,m}([a_2, \ldots, a_n])}$

For $m = 1$, we have the usual recurrence for continued fractions:

$$r_1([a_1, \ldots, a_n]) = a_1 + \frac{1}{r_1([a_2, \ldots, a_n])}$$

For $m = 2$, this is

$$r_{1,2}([a_1, \ldots, a_n]) = a_1 + \frac{r_{1,2}([a_2, \ldots, a_n])}{r_{2,2}([a_2, \ldots, a_n])}$$
$$r_{2,2}([a_1, \ldots, a_n]) = \left(\!\!\binom{a_1}{2}\!\!\right) + a_1 \frac{r_{1,2}([a_2, \ldots, a_n])}{r_{2,2}([a_2, \ldots, a_n])} + \frac{1}{r_{2,2}([a_2, \ldots, a_n])}$$

Alternate Statement of Theorem

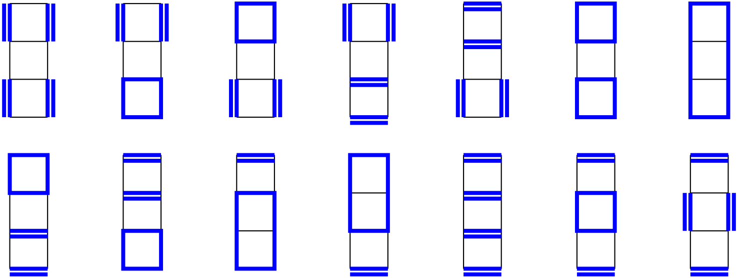
Theorem [Musiker, O, Schiffler, Zhang]

Let $x = [a_1, \dots, a_n]$. Then

$$r_{m,m}(x) = \frac{\text{\# of } m\text{-dimer covers of } G[a_1, \dots, a_n]}{\text{\# of } m\text{-dimer covers of } G[a_2, \dots, a_n]}$$

Example ($m = 2$)

$\frac{5}{2} = [2, 2]$, $\Lambda(2)\Lambda(2) = \begin{pmatrix} 14 & 8 & 3 \\ 8 & 5 & 2 \\ 3 & 2 & 1 \end{pmatrix}$, $r_{2,2}(5/2) = 14/3$



Extension to Real Numbers

Theorem [Musiker, O, Schiffler, Zhang]

Let $x = [a_1, a_2, \dots]$ be an irrational number, and let $x_n = [a_1, \dots, a_n]$ be its continuants. For fixed i and m, the sequence $r_{i,m}(x_n)$ converges.

Theorem [Musiker, O, Schiffler, Zhang]

If x is a quadratic irrational (i.e. its continued fraction is eventually periodic), then $r_{i,m}(x)$ is an algebraic number of degree (at most) $m + 1$.

An Example

Let G_n be the straight snake graph with $n-1$ squares.
The number of dimer covers of G_n are the Fibonacci numbers, and $\lim_{n\to\infty} \frac{f_n}{f_{n-1}} = \varphi = \frac{1+\sqrt{5}}{2}$.

Let ρ_m be the length of the longest diagonal in a regular $(2m+3)$-gon with side length 1. Then

$$r_{m,m}(\varphi) = \lim_{n\to\infty} \frac{\#\text{ of } m\text{-dimer covers of } G_n}{\#\text{ of } m\text{-dimer covers of } G_{n-1}} = \rho_m$$

For $m=2$, the number of double-dimer covers of G_n are 3, 6, 14, 31, $\ldots$, and the consecutive ratios converge to $\rho_2 = 4\cos^2(\pi/7) - 1 \approx 2.247$, whose minimal polynomial is $x^3 - 2x^2 - x + 1$.

Relation to Plane Partitions

There is a bijection between the set $\Omega_m(G)$ of m-dimer covers on a snake graph G, and reverse plane partitions on the dual snake graph G^*, with all parts at most m.

Stanley gave formulas for generating functions of reverse plane partitions.

Using Stanley's formulas, and considering G^* as a poset (and $\mathcal{L}(G^*)$ the set of linear extensions), we have:

$$\sum_{m\geq 0} |\Omega_m(G)|\ x^m = \frac{1}{(1-x)^N} \sum_{\sigma\in\mathcal{L}(G^*)} x^{\mathrm{des}(\sigma)}$$

A More Refined q-Version

The set $\Omega_m(G)$ has a natural partial order extending the one for ordinary dimer covers.

Replacing $|\Omega_m(G)|$ in the previous expression with the rank polynomial of $\Omega_m(G)$, we get:

$$\sum_{m\geq 0} \sum_{D\in\Omega_m(G)} q^{|D|} x^m = \frac{1}{(x;q)_N} \sum_{\sigma\in\mathcal{L}(G^*)} q^{\mathrm{maj}(\sigma)} x^{\mathrm{des}(\sigma)}$$

where $(x;q)_N = (1-x)(1-qx)(1-q^2x)\cdots(1-q^{N-1}x)$.

Example

For straight snake graphs, there is only one linear extension, and so:

$$\sum_{m\geq 0} \sum_{D\in\Omega_m(G)} q^{|D|} x^m = \frac{1}{(x;q)_N}$$

By the "q-binomial theorem", this means

$$\sum_{D\in\Omega_m(G)} q^{|D|} = \binom{N+m-1}{m}_q$$

Thank You!