Cluster Superalgebras from Triangulated Surfaces

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(joint with Gregg Musiker and Sylvester Zhang)
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Ptolemy’s Theorem

Take a quadrilateral inscribed in a circle, with lengths labelled as in the picture.

Then $xy = ac + bd$. 

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Then \( xy = ac + bd \).
Triangulated Polygons

Fix a triangulation. We can express any $x_{ij}$ in terms of $x$'s from the triangulation.

Example:

$x_{25} = x_{15}x_{23} + x_{12}x_{35}x_{13}$

$x_{13} = x_{14}x_{15}x_{23} + x_{12}x_{15}x_{34} + x_{12}x_{13}x_{45}$
For a polygon (inscribed in a circle), let

\[ x_{ij} = \text{length of diagonal } (i, j) \]
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**Example:**

\[
x_{25} = \frac{x_{15}x_{23} + x_{12}x_{35}}{x_{13}}
\]

\[
= x_{15}x_{23} + x_{12} \left( \frac{x_{15}x_{34} + x_{13}x_{45}}{x_{14}} \right)
\]

\[
= \frac{x_{13}}{x_{13}}
\]
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Fix a triangulation. We can express any \( x_{ij} \) in terms of \( x \)'s from the triangulation.

**Example:**

\[
\begin{align*}
x_{25} &= \frac{x_{15}x_{23} + x_{12}x_{35}}{x_{13}} \\
&= \frac{x_{15}x_{23} + x_{12} \left( \frac{x_{15}x_{34} + x_{13}x_{45}}{x_{14}} \right)}{x_{13}} \\
&= \frac{x_{14}x_{15}x_{23} + x_{12}x_{15}x_{34} + x_{12}x_{13}x_{45}}{x_{13}x_{14}}
\end{align*}
\]
Main Question

Question: How to predict what these expressions will look like after several iterations?

Answer: They are generating functions of "dimer covers" of certain graphs.
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Answer: They are generating functions of “dimer covers” of certain graphs.
A "dimer cover" (or "perfect matching") of a graph $\Gamma$ is a subset of edges so that every vertex is incident to one edge.

Examples:

If the edges have weights, then the "weight" of a dimer cover is the product of the edge weights.
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**Examples:**

$$\begin{array}{c}
\begin{array}{cccc}
& b & c & \\
\hline
a & & g & d \\
\hline
f & e & & \\
\end{array}
\end{array}$$

weight $= ace$

If the edges have weights, then the “weight” of a dimer cover is the product of the edge weights.
Snake Graphs

A "snake graph" is a planar graph built out of square tiles, where each tile is attached to the previous on either the right or top side.

Examples:

\[ W(G) = RR \]
\[ W(G) = URRU \]
\[ W(G) = RURUR \]

To each snake graph \( G \), we can associate a word \( W(G) \) in the alphabet \{R, U\} (for "right" and "up").
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**Examples:**

![Snake Graph Examples](image-url)
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Snake Graph from a Triangulation

Given a triangulated polygon, and a diagonal $\gamma$ which is not in the triangulation, we will construct a snake graph $G_{\gamma}$. Assume $\gamma$ crosses all interior edges of the triangulation (and all triangles).

Traverse $\gamma$
For each triangle (except first and last), look at its boundary side
For 2nd triangle, if right, label $R$, if left, label $U$
If same side, opposite letter $R$
If opposite side, same letter $W$

$(G_{\gamma}) = RRUU$

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\[ W(G_\gamma) = \text{RRUU} \]
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Snake Graph from a Triangulation

To label the snake graph, odd tiles match polygon labels, even tiles have opposite orientation.
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The Laurent Formula

Theorem \[\text{[Musiker, Schiffler]}\]

\[ x_\gamma = \frac{1}{\text{cross}(\gamma)} \sum_{M \in \mathcal{D}(G_\gamma)} \text{wt}(M) \]

where \(\text{cross}(\gamma)\) is the product of all edges of the triangulation which \(\gamma\) crosses.

Corollary
Each \(x_\gamma\) is a Laurent polynomial in the lengths of the diagonals from any fixed triangulation.

The Laurent Formula

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¹Gregg Musiker and Ralf Schiffler. “Cluster expansion formulas and perfect matchings”. In: Journal of Algebraic Combinatorics 32.2 (2010), pp. 187–209
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Example

\[ x_{25} = \frac{1}{x_{13}x_{14}} \left( x_{14}x_{23}x_{15} + x_{12}x_{34}x_{15} + x_{12}x_{13}x_{45} \right) \]
A "super algebra" is a $\mathbb{Z}_2$-graded algebra, i.e. $A = A_0 \oplus A_1$, (the "even" and "odd" parts) and $A_i A_j \subseteq A_{i+j}$. A super algebra is called "commutative" (or "super commutative") if for all $a, b \in A_0$ and $x, y \in A_1$:

$$ab = ba,\ ax = xa,\ xy = -yx$$

The basic example of a commutative super algebra is the one generated by $x_1, \ldots, x_n, \theta_1, \ldots, \theta_m$, subject to the relations $x_i x_j = x_j x_i$, $x_i \theta_j = \theta_j x_i$, $\theta_i \theta_j = -\theta_j \theta_i$ in particular, $\theta_2^2 = 0$. 

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Cluster Superalgebras  
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in particular, $\theta_i^2 = 0$
Given an $n$-gon, choose:

- a triangulation $T$
- an orientation of each edge in $T$ (We will not draw boundary orientations)

Consider the commutative super algebra with one even generator $x_{ij}$ for each diagonal in $T$, and one odd generator $\theta_{ijk}$ for each triangle in $T$.

The example above would have 7 even generators and 3 odd generators.
Given an \( n \)-gon, choose:

![Diagram of a pentagon with vertices labeled 1, 2, 3, 4, 5. ]
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Super Algebra from a Triangulation

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The example above would have 7 even generators and 3 odd generators.
The Super Ptolemy Relation

Given two adjacent triangles, we can \( \sqrt{f_{\text{lip}}} \) the diagonal:

\[
\begin{array}{cccc}
  a & b & c & d \\
  e & f & & \\
\end{array}
\]

\( \theta \quad \sigma \)

\[
\begin{array}{cccc}
  a & b & c & d \\
  & & & f \\
\end{array}
\]

\( \theta' \quad \sigma' \)

We define the new variables via the relations

1. \( ef = ac + bd + \sqrt{abcd} \)

2. \( \theta' = \sqrt{bd} \theta + \sqrt{ac} \sigma \)

3. \( \sqrt{ac} + \sqrt{bd} \sigma = \sqrt{bd} \sigma - \sqrt{ac} \theta \)

The Super Ptolemy Relation

Given two adjacent triangles, we can “flip” the diagonal:

\[
\begin{align*}
\text{original:} & & \\ a & \quad b & \quad c & \quad d & \text{ (orientation reversed)} \\
\theta & \quad \sigma & \quad & \quad e & \\
d & \quad & \quad & \quad & \sigma' \\
\text{new:} & & \\ a & \quad f & \quad b & \quad & \\
\theta' & \quad & \quad & \quad \sigma' & \\
d & \quad & \quad & \quad & \c
\end{align*}
\]

We define the new variables via the relations

\[ ef = ac + bd + \sqrt{abcd} \]

\[ \theta, \sigma = \sqrt{bd} \]

\[ \theta', \sigma' = \sqrt{bd} - \sqrt{ac} \]
The Super Ptolemy Relation

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We define the new variables via the relations:

\[ ef = ac + bd + \sqrt{abcd} \hat{\sigma} \theta \]
\[ \theta' = \sqrt{bd} \theta + \sqrt{ac} \sigma \]
\[ \sigma' = \sqrt{bd} \sigma - \sqrt{ac} \theta \]

---

The Super Ptolemy Relation

Given two adjacent triangles, we can “flip” the diagonal:

\[ a \quad \theta \quad b \]
\[ e \quad \sigma \quad f \]
\[ d \quad c \]

\[ \quad \rightarrow \quad \]

\[ a \quad f \quad b \]
\[ \theta' \quad \sigma' \quad \]
\[ d \quad c \]

(orientation reversed)

We define the new variables via the relations\(^1\):

\[ ef = ac + bd + \sqrt{abcd} \sigma \theta \]

---

The Super Ptolemy Relation

Given two adjacent triangles, we can "flip" the diagonal:

We define the new variables via the relations:\(^1\):

\[
ef = ac + bd + \sqrt{abcd} \sigma \theta
\]

\[
\theta' = \frac{\sqrt{bd} \theta + \sqrt{ac} \sigma}{\sqrt{ac + bd}}
\]

\[
\sigma' = \frac{\sqrt{bd} \sigma - \sqrt{ac} \theta}{\sqrt{ac + bd}}
\]

Another Set of Variables

Consider a triangle with vertices $i$, $j$, $k$, and associated variables $x_{ij}$, $x_{ik}$, $x_{jk}$, $\theta$:

\[ i \quad j \quad k \quad \theta \]

To each vertex of the triangle, define:

- An even variable $h_{ij} = x_{jk} x_{ik} x_{ij}$
- An odd variable $\theta(i) = \sqrt{h_{ij}}$
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Another Set of Variables

Consider a triangle with vertices $i, j, k$, and associated variables $x_{ij}, x_{ik}, x_{jk}, \theta$:

$$
\begin{align*}
\text{An even variable } & h_{ijk} := x_{jk} x_{ij} x_{ik} \\
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\end{align*}
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Another Set of Variables

Consider a triangle with vertices $i, j, k$, and associated variables $x_{ij}, x_{ik}, x_{jk}, \theta$:

To each vertex of the triangle, define:

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An odd variable $\theta(i) := \sqrt{h_{ikj}} \theta = \sqrt{x_{ij} x_{ik} x_{jk}} \theta$
Another Set of Variables

Consider a triangle with vertices $i, j, k$, and associated variables $x_{ij}, x_{ik}, x_{jk}, \theta$:

To each vertex of the triangle, define:

- An even variable $h_{jk}^i := \frac{x_{jk}}{x_{ij}x_{ik}}$
Another Set of Variables

Consider a triangle with vertices $i, j, k$, and associated variables $x_{ij}, x_{ik}, x_{jk}, \theta$:

To each vertex of the triangle, define:

- An even variable $h^i_{jk} := \frac{x_{jk}}{x_{ij}x_{ik}}$
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- An odd variable $\theta^{(i)} := \frac{\theta}{\sqrt{h^i_{jk}}} = \sqrt{\frac{x_{ij}x_{ik}}{x_{jk}}} \theta$
Using these new variables, the super Ptolemy relation looks like:

\[ f = \frac{ac + bd}{e} + \sigma(j)\theta(\ell) \]

\[ \varphi^{(i)} = \sigma^{(i)} + \theta^{(i)} \]

\[ \psi^{(k)} = \sigma^{(k)} - \theta^{(k)} \]
The Main Question

Starting with a fixed triangulation, we can reach any diagonal by a sequence of flips. Using the super Ptolemy relation, we will get some algebraic expression attached to this diagonal.

Question: Can we explicitly describe these algebraic expressions?

Answer: Yes! They are generating functions for "double dimer covers" of the snake graph.
The Main Question

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**Question:** Can we explicitly describe these algebraic expressions?

**Answer:** Yes! They are generating functions for “double dimer covers” of the snake graph.
A double dimer cover of a graph is the union of two dimer covers. It is composed of cycles and doubled edges.

Examples:
- Ovenhouse (Yale)
- Cluster Superalgebras
A *double dimer cover* of a graph is the union of two dimer covers. It is composed of cycles and doubled edges.
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Examples:
Double Dimer Covers on Snake Graphs

Every square face in a snake graph represents two triangles in the triangulation. We will label the faces with the odd variables of those triangles.

\[
\theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \quad \theta_5 \quad \theta_6
\]

The weight of a double dimer cover \( M \) is

\[
\text{wt}(M) = \prod_{\text{edges } e \in M} \sqrt{e} \cdot \prod_{\text{cycles } \theta_i \theta_j}
\]

where \( \theta_i, \theta_j \) are the labels in the bottom-left and top-right of the cycle.
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Double Dimer Covers on Snake Graphs

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Laurent Formula

Theorem

Given a fixed triangulation $\mathcal{T}$, the even variable $x_\gamma$ corresponding to a diagonal which is not in $\mathcal{T}$ is given by

$$x_\gamma = \sum_{\text{double dimers } M \in \text{DD}(G_\gamma)} \text{wt}(M)$$

Moreover, there is an ordering of the $\theta$'s which makes all terms positive.

Corollary ("Laurent Phenomenon")

Each $x_\gamma$ is a Laurent polynomial in the $\sqrt{x}_{ij}$'s and $\theta_{ijk}$'s.

Theorem [Musiker, O., Zhang]¹

Given a fixed triangulation $T$, the even variable $x_\gamma$ corresponding to a diagonal which is *not* in $T$ is given by

$$x_\gamma = \frac{1}{\text{cross}(\gamma)} \sum_{\text{double dimers } M \in DD(G_\gamma)} \text{wt}(M)$$

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Laurent Formula

**Theorem [Musiker, O., Zhang]¹**

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Each $x_\gamma$ is a Laurent polynomial in the $\sqrt{x_{ij}}$’s and $\theta_{ijk}$’s.

---

Example

\[ x_{14} \quad x_{45} \]
\[ \theta_1 \quad \theta_2 \quad \theta_3 \]
\[ x_{12} \]
\[ \theta_1 \quad \theta_2 \quad \theta_3 \]
\[ x_{23} \quad x_{13} \quad x_{15} \]
\[
x_{25} = \frac{1}{x_{13}x_{14}} \left( x_{14}x_{23}x_{15} + x_{12}x_{34}x_{15} + x_{12}x_{13}x_{45} \\
+ x_{15}\sqrt{x_{12}x_{14}x_{23}x_{34}} \theta_1 \theta_2 + x_{12}\sqrt{x_{13}x_{15}x_{34}x_{45}} \theta_2 \theta_3 \\
+ \sqrt{x_{12}x_{13}x_{14}x_{15}x_{23}x_{45}} \theta_1 \theta_3 \right)
\]
Fibonacci Numbers

Let $G_m$ be the snake graph which is a horizontal row of $m$ boxes:

The number of dimer covers of $G_m$ are the Fibonacci numbers.

<table>
<thead>
<tr>
<th>$D(G_1)$</th>
<th>$D(G_2)$</th>
<th>$D(G_3)$</th>
<th>$D(G_4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>
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Fibonacci Numbers

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\[ \begin{array}{cccc}
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Fibonacci Numbers

Let $G_m$ be the snake graph which is a horizontal row of $m$ boxes:

The number of dimer covers of $G_m$ are the Fibonacci numbers.

$|D(G_1)| = 2$

$|D(G_2)| = 3$

$|D(G_3)| = 5$

$|D(G_4)| = 8$
Geometric Interpretation

Rather than just polygons, we can triangulate any surface with boundary. For instance, a cylinder/annulus:

An arc connecting the two marked points corresponds to the snake graph $G_m$. 

\[
\begin{array}{c}
\text{Diagram}
\end{array}
\]
Geometric Interpretation

Rather than just polygons, we can triangulate any surface with boundary. For instance, a cylinder/annulus:

An arc connecting the two marked points corresponds to the snake graph $G_m$.

For the super version, add two odd variables ($\sigma$ and $\theta$) corresponding to the two triangles.
Consider the exterior algebra (over $\mathbb{R}$) with two generators $\sigma$ and $\theta$, and label the corners of faces of $G_m$ with $\sigma$ and $\theta$:

Note: For double dimer covers, a cycle of odd length has weight $\sigma \theta$. A cycle of even length has weight $\sigma^2 = 0$ or $\theta^2 = 0$. A double dimer with two (or more) cycles has weight 0.
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\[
\begin{array}{cccc}
\theta & \sigma & \theta & \sigma \\
\sigma & \theta & \sigma & \theta \\
\end{array}
\]

**Note:** For double dimer covers,
- A cycle of odd length has weight \( \sigma \theta \).
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![Diagram of labels on corners of faces]

**Note:** For double dimer covers,

- A cycle of odd length has weight $\sigma \theta$.
- A cycle of even length has weight $\sigma^2 = 0$ or $\theta^2 = 0$.
- A double dimer with two (or more) cycles has weight 0.
So the double dimer generating function is of the form
\[
\sum_{M \in \text{DD}} (G_m) \cdot \text{wt}(M) = x^m + y^m \sigma \theta
\]
where \(x^m\) is the number of double dimer covers using only doubled edges (these are the Fibonacci numbers)
\(y^m\) is the number of double dimer covers which have one cycle of odd length
Let \(\varepsilon := \sigma \theta\). Then we can think of this as an element of the ring of "dual numbers"
\(D := \mathbb{R}[\varepsilon]/(\varepsilon^2)\)
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Let $\varepsilon := \sigma \theta$. Then we can think of this as an element of the ring of "dual numbers"

$$\mathbb{D} := \mathbb{R}[\varepsilon]/(\varepsilon^2)$$
Dual Fibonacci Numbers

Define $F_0 = 1$, $F_1 = 1$, and for $n > 1$ $F_n :=$ $\sum_{M \in \mathcal{D}(G_n - 1)} \text{wt}(M) = x^{n-1} + y^{n-1} \varepsilon$

The first few terms are:

- $F_2 = 2 + \varepsilon$
- $F_3 = 3 + 2\varepsilon$
- $F_4 = 5 + 6\varepsilon$
- $F_5 = 8 + 12\varepsilon$
- $F_6 = 13 + 26\varepsilon$
- $F_7 = 21 + 50\varepsilon$
- $F_8 = 34 + 97\varepsilon$
Dual Fibonacci Numbers

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$$F_8 = 34 + 97\varepsilon$$
The dual Fibonacci numbers satisfy the recurrence:

\[ F_n = \begin{cases} 
(1 + \varepsilon)F_{n-1} + F_{n-2} & \text{if } n \text{ is even} \\
(1 + \varepsilon)F_{n-1} + F_{n-2} - \varepsilon & \text{if } n \text{ is odd} 
\end{cases} \]
Recurrences

The dual Fibonacci numbers satisfy the recurrence:

\[
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(1 + \varepsilon)F_{n-1} + F_{n-2} & \text{if } n \text{ is even} \\
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\end{cases}
\]

The Fibonacci numbers also satisfy:

\[
f_n = 3f_{n-2} - f_{n-4}
\]

The dual Fibonacci numbers satisfy

\[
F_n = \begin{cases} 
(3 + 2\varepsilon)F_{n-2} - F_{n-4} - \varepsilon & \text{if } n \text{ is even} \\
(3 + 2\varepsilon)F_{n-2} - F_{n-4} & \text{if } n \text{ is odd}
\end{cases}
\]
Generating Function

The generating function for the Fibonacci numbers is

\[ f(x) = \sum_{n=0}^{\infty} f_n x^n = 1 - x - x^2 \]

The corresponding generating function for dual Fibonacci numbers is

\[ F(x) = \sum_{n=0}^{\infty} F_n x^n = 1 - x - x^2 + x^2(1 - x^2)(1 - x - x^2)^2 \]

In other words,

\[ x^2(1 - x^2)(1 - x - x^2)^2 = \sum_{n=0}^{\infty} y_n x^n = x^2 + 2x^3 + 6x^4 + 12x^5 + 26x^6 + 50x^7 + \cdots \]

where \( y_n \) is the number of double dimer covers of \( G_n \) with a single odd-length cycle.
The generating function for the Fibonacci numbers is

\[ f(x) = \sum_{n=0}^{\infty} f_n x^n = \frac{1}{1 - x - x^2} \]
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In other words,

\[ \frac{x^2}{(1 - x^2)(1 - x - x^2)^2} = \sum_{n=0}^{\infty} y_n x^n \]

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Simpler Dual Fibonacci Numbers

Define another sequence $F_n$ by

$$F_n = \begin{cases} F_n & \text{if } n \text{ is odd} \\ F_n - \varepsilon & \text{if } n \text{ is even} \end{cases}$$

The first several terms are:

- $F_0 = 1 - \varepsilon$
- $F_1 = 1$
- $F_2 = 2$
- $F_3 = 3 + 2\varepsilon$
- $F_4 = 5 + 5\varepsilon$
- $F_5 = 8 + 12\varepsilon$
- $F_6 = 13 + 25\varepsilon$
Define another sequence $\mathcal{F}_n$ by

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The first several terms are:

\[ \begin{align*}
F_0 &= 1 - \varepsilon \\
F_1 &= 1 \\
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F_3 &= 3 + 2\varepsilon \\
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F_4 &= 8 + 12\varepsilon \\
F_5 &= 13 + 25\varepsilon
\end{align*} \]
Simpler Dual Fibonacci Numbers

This satisfies the recurrence

\[ F_n = (1 + \epsilon) F_{n-1} + F_{n-2} \]

The generating function is

\[ \sum_{n=0}^{\infty} F_n x^n = \frac{1}{1 - (1 + \epsilon)x - x^2} \]
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The generating function is

\[
\sum_{n=0}^{\infty} F_n x^n = \frac{1 - \varepsilon}{1 - (1 + \varepsilon)x - x^2}
\]
Continued Fractions

The continued fraction expansion of the ratio of Fibonacci numbers is

\[
\frac{f_n}{f_{n-1}} = [1, 1, 1, \ldots, 1] = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}
\]

Examples:

\[
\frac{3}{2} = 1 + \frac{1}{1 + \frac{1}{1}}
\]

\[
\frac{5}{3} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}
\]

\[
\frac{8}{5} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}
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The continued fraction expansion of the ratio of Fibonacci numbers is

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\[
\frac{8}{5} = \frac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{1}{1}}}}
\]
Continued Fractions

There is a similar statement for the $F_n$ sequence:

$$F_n - 1 = \left[ 1 + \varepsilon, 1 + \varepsilon, ..., 1 + \varepsilon \right]$$

Examples:

$$3 + 2\varepsilon = (1 + \varepsilon) + \left( 1 + \varepsilon \right) + \left( 1 + \varepsilon \right)$$

$$5 + 5\varepsilon = (1 + \varepsilon) + \left( 1 + \varepsilon \right) + \left( 1 + \varepsilon \right) + \left( 1 + \varepsilon \right) + \left( 1 + \varepsilon \right)$$

$$8 + 12\varepsilon = (1 + \varepsilon) + \left( 1 + \varepsilon \right) + \left( 1 + \varepsilon \right) + \left( 1 + \varepsilon \right) + \left( 1 + \varepsilon \right) + \left( 1 + \varepsilon \right) + \left( 1 + \varepsilon \right)$$
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Continued Fractions

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$$\frac{F_n}{F_{n-1}} = [1 + \varepsilon, 1 + \varepsilon, \ldots, 1 + \varepsilon]$$

Examples:

$$\frac{3 + 2\varepsilon}{2} = (1 + \varepsilon) + \frac{1}{(1 + \varepsilon) + \frac{1}{(1+\varepsilon)}}$$

$$\frac{5 + 5\varepsilon}{3 + 2\varepsilon} = (1 + \varepsilon) + \frac{1}{(1 + \varepsilon) + \frac{1}{(1+\varepsilon) + \frac{1}{(1+\varepsilon)}}}$$

$$\frac{8 + 12\varepsilon}{5 + 5\varepsilon} = (1 + \varepsilon) + \frac{1}{(1 + \varepsilon) + \frac{1}{(1+\varepsilon) + \frac{1}{(1+\varepsilon) + \frac{1}{(1+\varepsilon)}}}}$$
Thank You!