Cluster Superalgebras from Triangulated Surfaces

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(joint with Gregg Musiker and Sylvester Zhang)
ArXiv: 2102.09143, 2110.06497, and 2208.13664

Yale University

September, 2022
UConn Algebra Seminar

Ptolemy's Theorem

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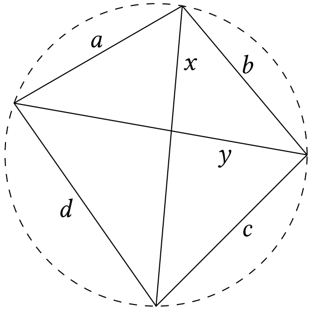
Take a quadrilateral inscribed in a circle, with lengths labelled as in the picture.



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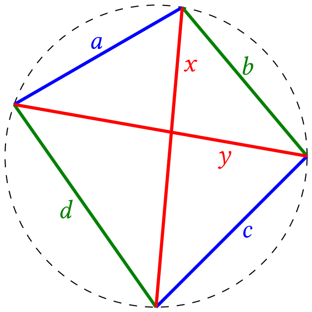

Then $xy = ac + bd$.

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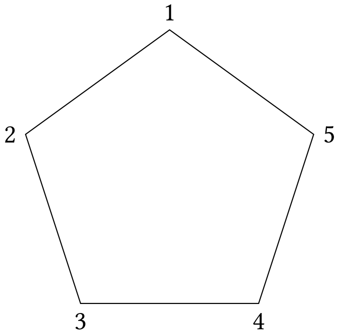

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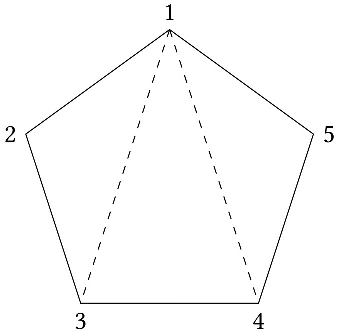

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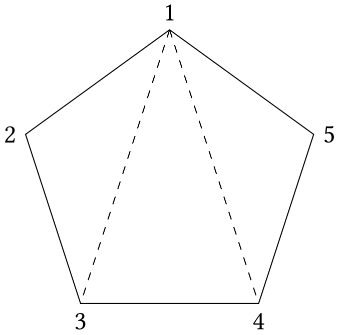

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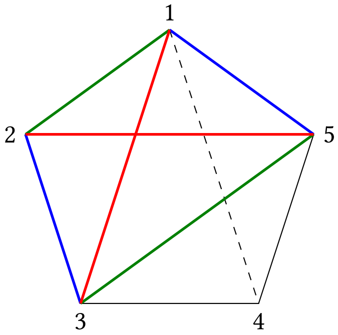

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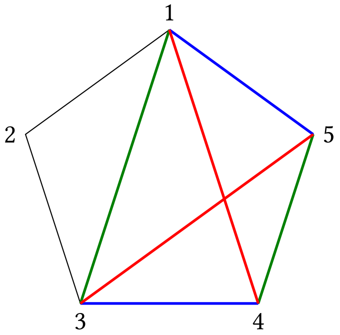

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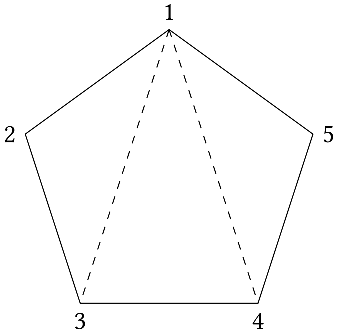

Main Question

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Answer: They are generating functions of “*dimer covers*” of certain graphs.

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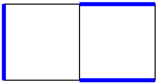

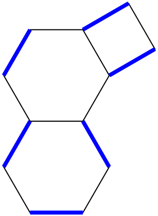

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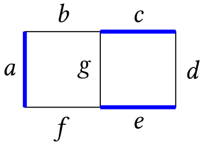
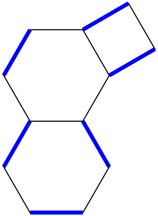


weight $= ace$



If the edges have weights, then the "*weight*" of a dimer cover is the product of the edge weights.

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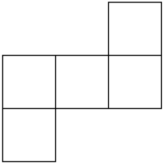

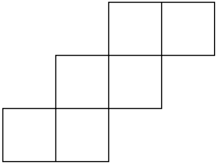

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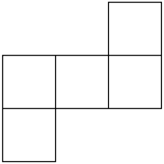
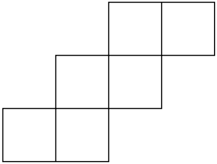
Examples:

$W(G) = RR$



$W(G) = URRU$



$W(G) = RURUR$

To each snake graph G, we can associate a word $W(G)$ in the alphabet $\{R, U\}$ (for "*right*" and "*up*").

Snake Graph from a Triangulation

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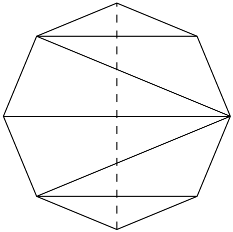

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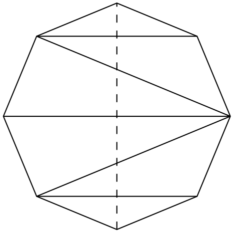

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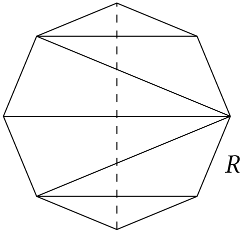

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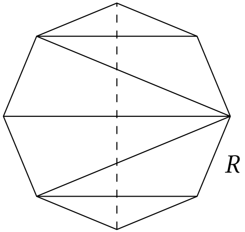

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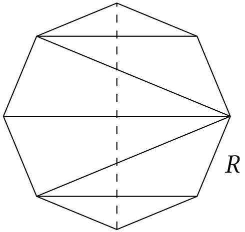

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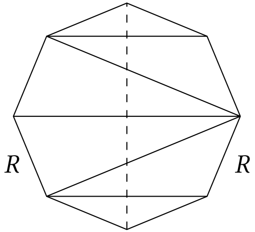

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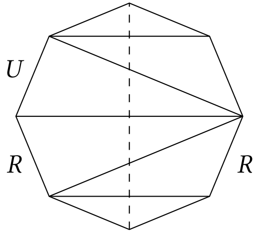

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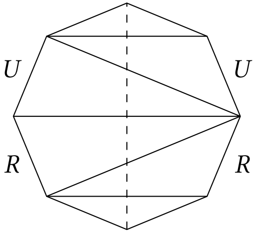

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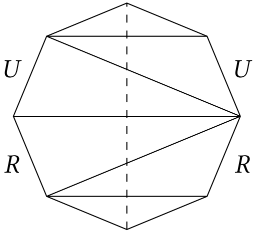

$$W(G_\gamma) = RRUU$$

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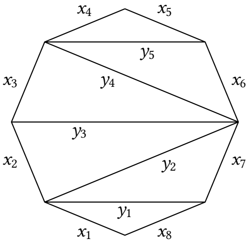

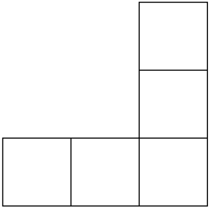

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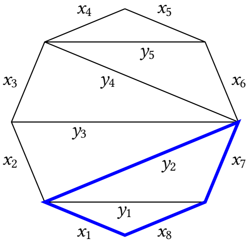

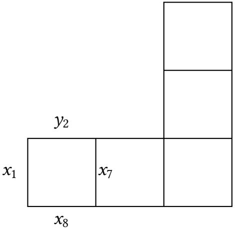

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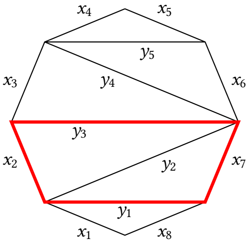

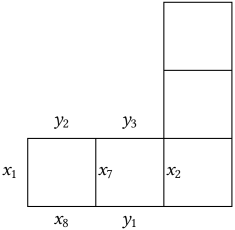

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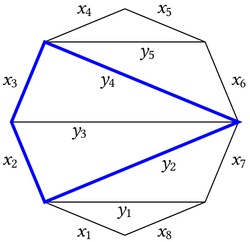

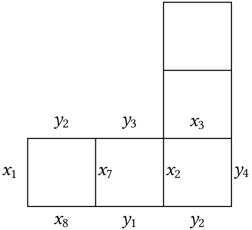

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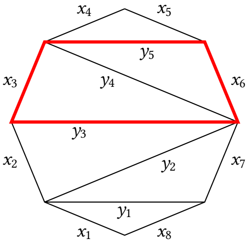

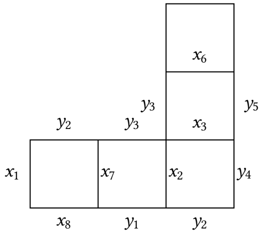

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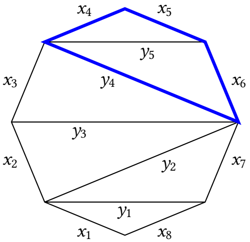

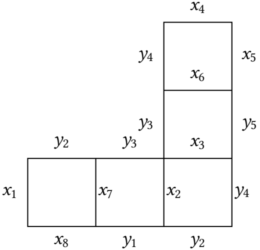

The Laurent Formula

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Theorem [Musiker, Schiffler][1]

$$x_\gamma = \frac{1}{\text{cross}(\gamma)} \sum_{M \in D(G_\gamma)} \text{wt}(M)$$

where $\text{cross}(\gamma)$ is the product of all edges of the triangulation which γ crosses.

[1] Gregg Musiker and Ralf Schiffler. "Cluster expansion formulas and perfect matchings". In: *Journal of Algebraic Combinatorics* 32.2 (2010), pp. 187–209

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Corollary

Each x_γ is a Laurent polynomial in the lengths of the diagonals from any fixed triangulation.

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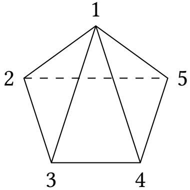
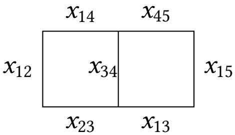
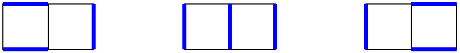
Example







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Super Algebras

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The basic example of a commutative super algebra is the one generated by $x_1, \dots, x_n, \theta_1, \dots, \theta_m$, subject to the relations

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in particular, $\theta_i^2 = 0$

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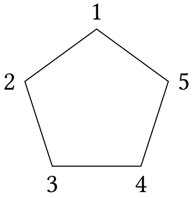

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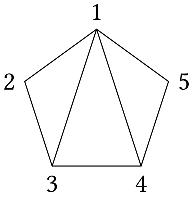

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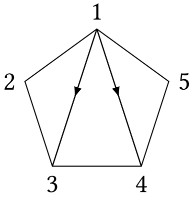
  (We will not draw boundary orientations)



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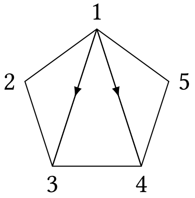
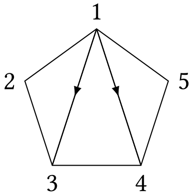


Consider the commutative super algebra with one even generator x_{ij} for each diagonal in T, and one odd generator θ_{ijk} for each triangle in T.

Super Algebra from a Triangulation

Given an n-gon, choose:

- a triangulation T
- an orientation of each edge in T
  (We will not draw boundary orientations)



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The example above would have 7 even generators and 3 odd generators.

The Super Ptolemy Relation

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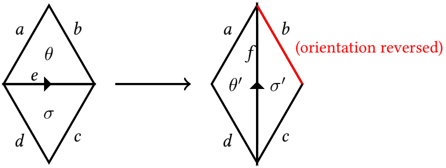
Given two adjacent triangles, we can "*flip*" the diagonal:



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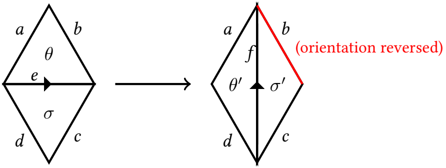

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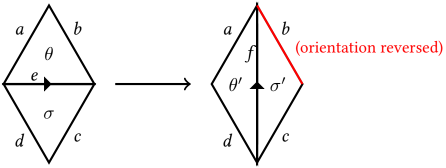

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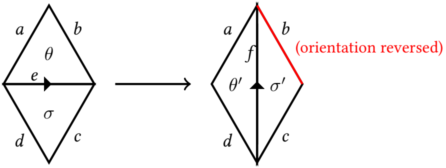

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$$\theta' = \frac{\sqrt{bd}\,\theta + \sqrt{ac}\,\sigma}{\sqrt{ac+bd}}$$

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Another Set of Variables

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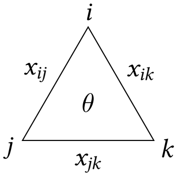
Consider a triangle with vertices i, j, k,
and associated variables $x_{ij}, x_{ik}, x_{jk}, \theta$:



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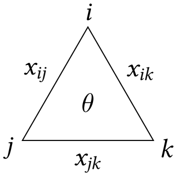

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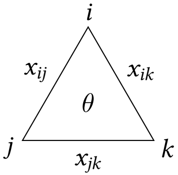

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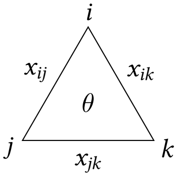

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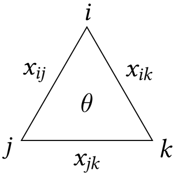

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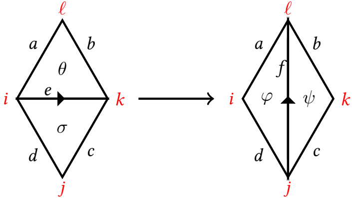
Alternate Super Ptolemy Relation



Using these new variables, the super Ptolemy relation looks like:

$$f = \frac{ac + bd}{e} + \sigma_{(j)}\theta_{(\ell)}$$

$$\varphi^{(i)} = \sigma^{(i)} + \theta^{(i)}$$

$$\psi^{(k)} = \sigma^{(k)} - \theta^{(k)}$$

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Question: Can we explicitly describe these algebraic expressions?

Answer: Yes! They are generating functions for "*double dimer covers*" of the snake graph.

Double Dimer Covers

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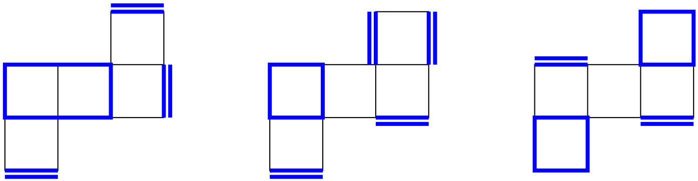
Examples:



Double Dimer Covers on Snake Graphs

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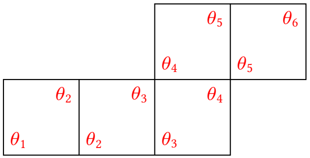
Every square face in a snake graph represents two triangles in the triangulation. We will label the faces with the odd variables of those triangles.



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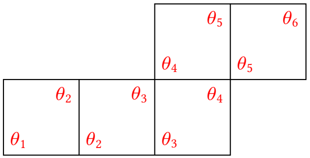

The *weight* of a double dimer cover M is

$$\mathrm{wt}(M) = \left(\prod_{\substack{\text{edges} \\ e \in M}} \sqrt{e} \right) \cdot \left(\prod_{\text{cycles}} \theta_i \theta_j \right)$$

where θ_i, θ_j are the labels in the bottom-left and top-right of the cycle.

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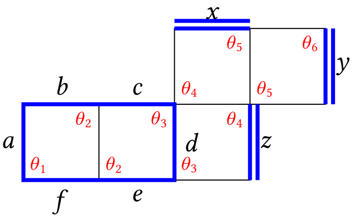

$$\text{weight} = xyz\sqrt{abcdef}\,\theta_1\theta_3$$

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Laurent Formula

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Theorem[Musiker, O., Zhang][1]

Given a fixed triangulation T, the even variable x_γ corresponding to a diagonal which is *not* in T is given by

$$x_\gamma = \frac{1}{\mathrm{cross}(\gamma)} \sum_{\substack{\text{double dimers} \\ M \in DD(G_\gamma)}} \mathrm{wt}(M)$$

Moreover, there is an ordering of the θ's which makes all terms positive.

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Corollary ("Laurent Phenomenon")

Each x_γ is a Laurent polynomial in the $\sqrt{x_{ij}}$'s and θ_{ijk}'s.

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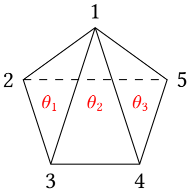
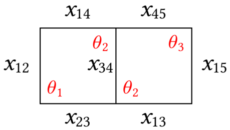
Example


1
2
5
3
4
θ_1
θ_2
θ_3


x_{14}
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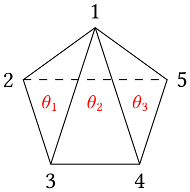

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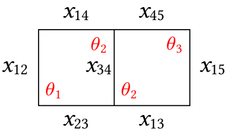

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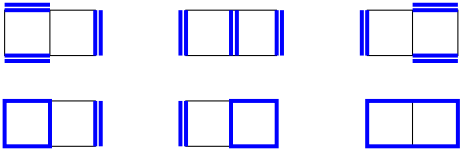

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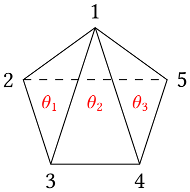

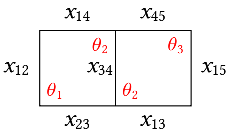

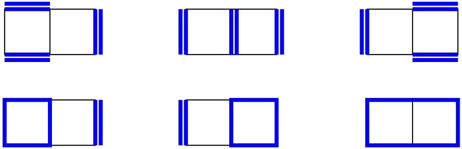

$$x_{25} = \frac{1}{x_{13}x_{14}}\Big(x_{14}x_{23}x_{15} + x_{12}x_{34}x_{15} + x_{12}x_{13}x_{45}$$
$$+ x_{15}\sqrt{x_{12}x_{14}x_{23}x_{34}}\,\theta_1\theta_2 + x_{12}\sqrt{x_{13}x_{15}x_{34}x_{45}}\,\theta_2\theta_3$$
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The number of dimer covers of G_m are the Fibonacci numbers.

$|D(G_1)| = 2$ 

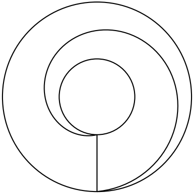
$|D(G_2)| = 3$ 

$|D(G_3)| = 5$ 

$|D(G_4)| = 8$ 

Geometric Interpretation

Rather than just polygons, we can triangulate any surface with boundary. For instance, a cylinder/annulus:



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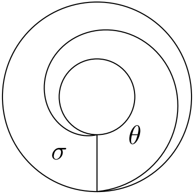

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For the super version, add two odd variables (σ and θ) corresponding to the two triangles.

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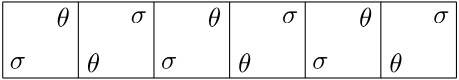

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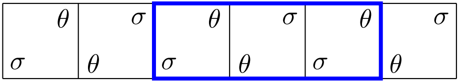

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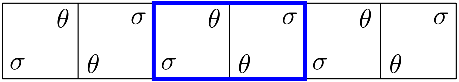

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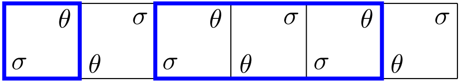

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Let $\varepsilon := \sigma\theta$. Then we can think of this as an element of the ring of "*dual numbers*"

$$\mathbb{D} := \mathbb{R}[\varepsilon]/(\varepsilon^2)$$

Dual Fibonacci Numbers

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Define $F_0 = 1$, $F_1 = 1$, and for $n > 1$

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The first few terms are

$$\begin{aligned} F_2 &= 2 + \varepsilon \\ F_3 &= 3 + 2\varepsilon \\ F_4 &= 5 + 6\varepsilon \\ F_5 &= 8 + 12\varepsilon \\ F_6 &= 13 + 26\varepsilon \\ F_7 &= 21 + 50\varepsilon \\ F_8 &= 34 + 97\varepsilon \end{aligned}$$

Recurrences

The dual Fibonacci numbers satisfy the recurrence:

$$F_n = \begin{cases} (1+\varepsilon)F_{n-1} + F_{n-2} & \text{if } n \text{ is even} \\ (1+\varepsilon)F_{n-1} + F_{n-2} - \varepsilon & \text{if } n \text{ is odd} \end{cases}$$

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The Fibonacci numbers also satisfy:

$$f_n = 3f_{n-2} - f_{n-4}$$

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In other words,

$$\begin{aligned} \frac{x^2}{(1 - x^2)(1 - x - x^2)^2} &= \sum_{n=0}^{\infty} y_n x^n \\ &= x^2 + 2x^3 + 6x^4 + 12x^5 + 26x^6 + 50x^7 + \cdots \end{aligned}$$

where y_n is the number of double dimer covers of G_n with a single odd-length cycle.

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Define another sequence $\mathcal{F}_n$ by

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The first several terms are:

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$$\mathcal{F}_4 = 8 + 12\varepsilon$$
$$\mathcal{F}_5 = 13 + 25\varepsilon$$

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The generating function is

$$\sum_{n=0}^{\infty} \mathcal{F}_n x^n = \frac{1-\varepsilon}{1-(1+\varepsilon)x - x^2}$$

Continued Fractions

Continued Fractions

The continued fraction expansion of the ratio of Fibonacci numbers is

$$\frac{f_n}{f_{n-1}} = [1, 1, \ldots, 1] = 1 + \frac{1}{1 + \frac{1}{1+\cdots}}$$

Continued Fractions

The continued fraction expansion of the ratio of Fibonacci numbers is

$$\frac{f_n}{f_{n-1}} = [1, 1, \ldots, 1] = 1 + \frac{1}{1 + \frac{1}{1+\cdots}}$$

Examples:

$$\frac{3}{2} = 1 + \frac{1}{1 + \frac{1}{1}}$$

$$\frac{5}{3} = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1}}}$$

$$\frac{8}{5} = \frac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{1}{1}}}}$$

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There is a similar statement for the $\mathcal{F}_n$ sequence:

$$\frac{\mathcal{F}_n}{\mathcal{F}_{n-1}} = [1+\varepsilon,\ 1+\varepsilon,\ \dots,\ 1+\varepsilon]$$

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$$\frac{\mathcal{F}_n}{\mathcal{F}_{n-1}} = [1+\varepsilon,\ 1+\varepsilon,\ \ldots,\ 1+\varepsilon]$$

Examples:

$$\frac{3+2\varepsilon}{2} = (1+\varepsilon) + \cfrac{1}{(1+\varepsilon) + \frac{1}{(1+\varepsilon)}}$$

$$\frac{5+5\varepsilon}{3+2\varepsilon} = (1+\varepsilon) + \cfrac{1}{(1+\varepsilon) + \frac{1}{(1+\varepsilon)+\frac{1}{(1+\varepsilon)}}}$$

$$\frac{8+12\varepsilon}{5+5\varepsilon} = (1+\varepsilon) + \cfrac{1}{(1+\varepsilon) + \frac{1}{(1+\varepsilon)+\frac{1}{(1+\varepsilon)+\frac{1}{(1+\varepsilon)}}}}$$

Thank You!