Lecture #24 10/26/2016

· Last time: "Simple Interest"

Initial data:

- Lender (L) lends \$P to Borrower (B)
- interest rate (r) ← use APR when it is a rate per year.
- term (t) - length of time

At the end the Borrower (B) has to pay \$F back to Lender (L)

Simple Interest Formula: $\boxed{F = P + P\cdot t\cdot r = P\cdot(1+t\cdot r)}$

Another way to write it is via $I := F - P$.

$$\boxed{I = t\cdot r\cdot P}$$

Ex1: Alice bought a 5-year government bond at APR = 4% with a face value (= \$-amount she will get back after 5 years) of \$6,000. How much did she pay originally?

$$\left.\begin{array}{l} F = P\cdot(1+t\cdot r) \Rightarrow P = \frac{F}{1+t\cdot r} \\ F = \$6{,}000,\ r = \frac{4}{100},\ t = 5 \end{array}\right\} \Rightarrow P = \frac{\$6{,}000}{1+\frac{1}{5}} = \frac{\$6{,}000}{\frac{6}{5}} = \$5{,}000$$

Answer: \$5,000 is the amount Alice spent on this bond orig.

Ex2: John took a loan from a bank at the amount of \$4000 for 5 years with simple interest. At the end he paid back to the bank \$5000. Find the APR.

$$\left.\begin{array}{l} F = P(1+t\cdot r) \Rightarrow 1+t\cdot r = \frac{F}{P} \\ F = \$5000,\ P = \$4000,\ t = 5 \end{array}\right\} \Rightarrow 1+5r = \frac{5000}{4000} \Rightarrow 5r = \frac{1}{4} \Rightarrow r = \frac{1}{20}$$

As $\frac{1}{20} = 0.05 \Rightarrow$ APR was 5%

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- Today: "Compound Interest" (§10.3)

Basic Idea: Previously accumulated interest generates an interest

Compound Interest Formula: $\boxed{F = P\cdot(1+r)^t}$

Let us consider the same example as at the end of last lecture, but in the setting of compound interest.

Ex3: Bob took a loan of \$6000 at compound annual interest at APR = 5% and for the length of 4 years. What will he pay at the end?

- First, at the moment he borrows money, he owes \$6,000
- After 1^{st} year, he owes $\$6,000 + \underbrace{(5\% \text{ of } \$6000)}_{300\,\$} = \$6,300$
- After 2^{nd} Year, he owes $\$6,300 + \underbrace{(5\% \text{ of } \$6,300)}_{6300\cdot\frac{5}{100} = 315} = \$6,615$
- After 3^{d} Year, he owes $\$6,615 + \underbrace{(5\% \text{ of } \$6,615)}_{6615\cdot\frac{5}{100} = 330^{75}} = \$6,945^{75}$
- Finally after 4^{th} Year, he owes $\underbrace{\$6,945^{75} + \underbrace{(5\% \text{ of } 6,945^{75})}_{\approx 347^{29}}}_{\approx \$7293^{04}}$

Bob has to pay back $\$7293^{04}$, while as we saw last time at simple interest he would have to pay back only \$7200

At this example, the difference is not really big.

Rmk But if we change the length of time to e.g. 20 years, then at simple interest he would have to pay back \$12,000, while at compound interest - $\$15,919^{79}$

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Ex 4: John invested \$2000 into a CD (certificate of deposit) with an APR of 3.6% compounded annually. Find the future value (= cash value of CD at the end of its term) corresponding to $t = 1, 5, 10, 15, 20$.

- $t=1$: $\$2000 \cdot \left(1 + \frac{3.6}{100}\right) = \$2072$
- $t=5$: $\$2000 \cdot \left(1 + \frac{3.6}{100}\right)^5 \approx \$2{,}386^{87}$
- $t=10$: $\$2000 \cdot \left(1 + \frac{3.6}{100}\right)^{10} \approx \$2{,}848^{57}$
- $t=15$: $\$2000 \cdot \left(1 + \frac{3.6}{100}\right)^{15} \approx \$3{,}399^{59}$
- $t=20$: $\$2000 \cdot \left(1 + \frac{3.6}{100}\right)^{20} \approx \$4{,}057^{13}$

Rmk (Rule of 72): It takes approximately $\frac{72}{APR}$ to double your original investment.

[in the above example, $\frac{72}{3.6} = 20$ and we saw that future value after 20 years is almost twice original value].

- General Compounding

There is no reason to assume that interest is compounded only once per year. In the example of CD's accounts, there are some with interest compounded semi-annualy, quarterly, or montly.

General compounding f-la : $\boxed{F = P \cdot (1+p)^T}$ - same as before

! If it is e.g. a quarterly compound interest with APR = 3.6%, then it means the interest per quarter is $\frac{1}{4} \cdot 3.6\% = 0.9\%$.

Ex 5: Consider the setting of Ex 4, but with interest compounded on the monthly basis. Find the future values after 1, 5, 10, 15, 20 years.

► Since APR = 3.6% and one year consists of 12 months, it means that monthly interest is $\frac{3.6\%}{12} = 0.3\%$.

$t = 1\text{ year} = 12\text{ months} \rightsquigarrow 2000 \cdot \left(1+\frac{0.3}{100}\right)^{12} \approx 2073^{20}$

$t = 5\text{ years} = 60\text{ months} \rightsquigarrow 2000 \cdot \left(1+\frac{0.3}{100}\right)^{60} \approx 2393^{79}$

$t = 10\text{ years} = 120\text{ months} \rightsquigarrow 2000 \cdot \left(1+\frac{0.3}{100}\right)^{120} \approx 2865^{11}$

$t = 15\text{ years} = 180\text{ months} \rightsquigarrow 2000 \cdot \left(1+\frac{0.3}{100}\right)^{180} \approx 3429^{24}$

$t = 20\text{ years} = 240\text{ months} \rightsquigarrow 2000 \cdot \left(1+\frac{0.3}{100}\right)^{240} \approx 4104^{44}$ ∎

Rmk: After all, the difference is not that big!

• Continuous Compounding

We started originally from the case, when interest was compounded once a year, but then also discussed that often it is compounded once per a shorter period of time. In the "limit", we get a situation when interest is compounded continuously

Q: How do we compute the final value?

Continuous Compounding F-la: $\boxed{F = P \cdot e^{rt}}$

! here $e \approx 2.718281828\ldots$ - Euler's Number