# Lecture #2

09/05/2017

- Organization
  - Ask if Tu, We $2^{00}-3^{30}$ is a good time for office hours.
  - Remind that the $1^{st}$ homework is due Thur, Sept. 7
    ↳ The students should start working on it right away so that if some questions arise, these can be discussed during Wednesday office hours.

## Reminder of last lecture

- Ask if there are some questions from last lecture.
- Last time we discussed:
  - computing the distance between 2 points in $\mathbb{R}^2$ and $\mathbb{R}^3$
  - equation of a sphere, completing squares to find center & radius
  - vectors in $\mathbb{R}^2$, $\mathbb{R}^3$ and two operations with them: addition, multiplication by a scalar
  - components of vectors (via a choice of coordinate system).
  - dot product

Let us warm up by doing a couple of examples, relevant to Homework #1

Ex1: Find the lengths of the sides of the triangle PQR with $P(1,2,3)$, $Q(3,1,2)$, $R(0,1,2)$.
Is it a right triangle? Is it an isosceles triangle?

Ex2: Find an equation of the sphere passing through $(4,5,3)$ and whose center is $(2,1,0)$.

Recall that dot product of two vectors $\vec{a}=\langle a_1,a_2,a_3\rangle$, $\vec{b}=\langle b_1,b_2,b_3\rangle$ is a number defined as $\vec{a}\cdot\vec{b}:=a_1b_1+a_2b_2+a_3b_3$. One of the reasons why it is useful is due to the equality

(1) $\boxed{\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\theta,\ \theta\text{ - angle between } \vec{a} \text{ and } \vec{b}}$

Ex3: Determine the angle between vectors $\vec{a}=\langle 3,0,-4\rangle$ and $\vec{b}=\langle 2,3,1\rangle$

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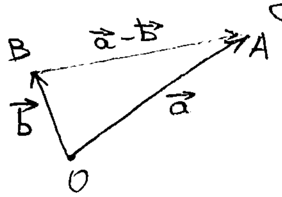
**Explanation of (1)** (maybe skipped in the class)

First we note that if $\vec{a} = \langle a_1, a_2, a_3 \rangle$, then $\vec{a}\cdot\vec{a} = a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2$.

Next, we consider the following picture




$$|AB|^2 = |\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} = |OA|^2 + |OB|^2 - 2\,\vec{a}\cdot\vec{b}$$

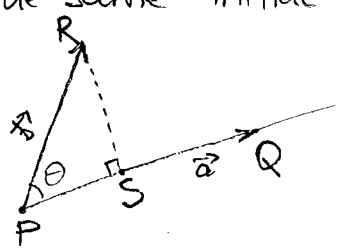
However, $|AB|^2 = |OA|^2 + |OB|^2 - 2\cdot|OA|\cdot|OB|\cdot\cos\theta$ by the Law of cosines

$$\Rightarrow \vec{a}\cdot\vec{b} = |OA|\cdot|OB|\cdot\cos\theta = |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta$$

Rmk: $\vec{a}$ is perpendicular to $\vec{b}$ iff $\vec{a}\cdot\vec{b} = 0$.

## Projections (end of Section 12.3)

Consider representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors $\vec{a}$ and $\vec{b}$ with the same initial point.




Let S be the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$.

**Definition 1:** The vector $\overrightarrow{PS}$ is called the *vector projection* of $\vec{b}$ onto $\vec{a}$ and is denoted by $\text{proj}_{\vec{a}}\vec{b}$.

**Definition 2:** The *scalar projection* of $\vec{b}$ onto $\vec{a}$ is the signed magnitude of the vector projection, i.e. the number $|\vec{b}|\cdot\cos\theta$ and is denoted by $\text{comp}_{\vec{a}}\vec{b}$.

Rmk: $\text{comp}_{\vec{a}}\vec{b} > 0$ iff $0 < \theta < \pi/2 \,(=90^\circ)$

$\text{comp}_{\vec{a}}\vec{b} = 0$ iff $\theta = \pi/2 \,(=90^\circ)$

$\text{comp}_{\vec{a}}\vec{b} < 0$ iff $\theta > \pi/2 \,(=90^\circ)$

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**Question:** How do we compute $\text{proj}_{\vec{a}}\vec{b}$ and $\text{comp}_{\vec{a}}\vec{b}$?

- Recall $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta \Rightarrow |\vec{b}|\cdot\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

  So: $\boxed{\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$ (note: $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \underbrace{\dfrac{\vec{a}}{|\vec{a}|}}_{\text{unit vector in the direction of } \vec{a}}\cdot\vec{b}$)

- As we know the direction and the "oriented magnitude", we can immediately find $\text{proj}_{\vec{a}}\vec{b}$ by multiplying a unit vector in the direction of $\vec{a}$ (equal to $\frac{\vec{a}}{|\vec{a}|}$) by $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$)

  So: $\boxed{\text{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\cdot\vec{a}}$

**Ex 4:** Find the scalar and vector projections of $\vec{b}$ onto $\vec{a}$:

(i) $\vec{a} = 3\vec{i} - 3\vec{j} + 2\vec{k}$, $\vec{b} = 2\vec{i} + 5\vec{j} - \vec{k}$

(ii) $\vec{a} = \langle -6, 10\rangle$, $\vec{b} = \langle 3, 6\rangle$.

**Ex 5*:** Suppose that $\vec{a}$ and $\vec{b}$ are nonzero vectors.

(a) Under what circumstances is $\text{comp}_{\vec{a}}\vec{b} = \text{comp}_{\vec{b}}\vec{a}$ ?

(b) Under what circumstances is $\text{proj}_{\vec{a}}\vec{b} = \text{proj}_{\vec{b}}\vec{a}$ ?

- Cross Product (Section 12.4)

Given two vectors $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$, one can define their cross product, denoted $\vec{a}\times\vec{b}$ as a vector in $\mathbb{R}^3$ given by

$$\boxed{\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle}$$

**Rmk:** Unlike the dot product, the cross product is defined only for $\vec{a}, \vec{b}$ being the 3-dimensional vectors.

To make this f-la easier to remember, one uses the determinants

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \overset{\text{definition}}{:=} \vec{i}\cdot\underbrace{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}}_{a_2b_3 - a_3b_2} - \vec{j}\cdot\underbrace{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}}_{a_1b_3 - a_3b_1} + \vec{k}\underbrace{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}_{a_1b_2 - a_2b_1}$$

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The importance of $\vec{a}\times\vec{b}$ is due to the following result:

**Theorem**: The vector $\vec{a}\times\vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$.

(see p. 816 of your textbook for a proof).

Moreover, the direction of $\vec{a}\times\vec{b}$ is given by the "right-hand rule": if the fingers of your right hand curl in the direction of a rotation (through an angle $\theta < 180°$) from $\vec{a}$ to $\vec{b}$, then your thumb points in the direction of $\vec{a}\times\vec{b}$.

Ex 6: Find the cross-product and verify that it is orthogonal to both $\vec{a}$ and $\vec{b}$ in the following cases:

(i) $\vec{a} = \langle 1, 4, -1\rangle$, $\vec{b} = \langle 2, -1, 3\rangle$

(ii) $\vec{a} = 2\vec{i} - \vec{j} + 3\vec{k}$, $\vec{b} = -\vec{i} - 2\vec{j} + \vec{k}$

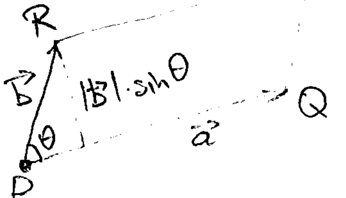
The geometric meaning of the magnitude $|\vec{a}\times\vec{b}|$ is as follows:

$$\boxed{|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin\theta, \quad \theta\text{-angle between } \vec{a} \text{ and } \vec{b}}$$

(see Theorem 9 and its proof on p. 817 of your textbook).

Cor.: $\vec{a}\times\vec{b} = 0$ iff $\vec{a}$ is parallel to $\vec{b}$.

Another geometric interpretation of $|\vec{a}\times\vec{b}|$ is as follows:




Consider the parallelogram with vectors $\vec{PQ} = \vec{a}$, $\vec{PR} = \vec{b}$. Then:

$$\text{Area}(PQTR) =: A = |\vec{a}|\cdot(|\vec{b}|\cdot\sin\theta) = |\vec{a}\times\vec{b}|.$$

So: The magnitude of $\vec{a}\times\vec{b}$ is equal to the area $A$ of parallelogr.

$\Downarrow$

The area of the triangle $\Delta PQR$ is equal to $\frac{1}{2}|\vec{PQ}\times\vec{PR}|$

**Ex 7**: Find the area of the triangle with vertices
$P(1,0,-1)$, $Q(2,1,-3)$, $R(3,-1,5)$

Finally, as pointed in the beginning, the cross product of two vectors is perpendicular to each of them.

**Ex 8**: (i) Find a vector perpendicular to the plane that passes through the points $P(1,0,-1)$, $Q(2,1,-3)$, $R(3,-1,5)$

(ii) Find two unit vectors orthogonal to both $\langle 1,1,1\rangle$ and $\langle 2,-3,-1\rangle$.

**Warning**: The cross product is neither commutative nor associative, i.e.

$$\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$$

$$(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$$

**However**: $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ and this number is called the scalar triple product of the vectors $\vec{a},\vec{b},\vec{c}$.

It can be written as a determinant of a $3\times 3$ matrix:

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

**Geometric meaning**: The absolute value $|\vec{a}\cdot(\vec{b}\times\vec{c})|$ equals the volume of the parallelepiped determined by $\vec{a},\vec{b},\vec{c}$.